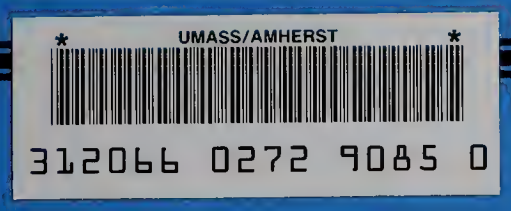

Mathematics Curriculum Framework

PUBLIC COMMENT DRAFT

September 28, 1999

Massachusetts Department of Education

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Mr. Hillel Bromberg

To: Interested Parties
From: David P. Driscoll, Commissioner of Education
Re: Mathematics Framework Draft
Date: September 28, 1999

I am pleased to present for your comment a draft of the revised Mathematics Curriculum Framework. In accordance with the Massachusetts Education Reform Act of 1993, the Board and the Department of Education are committed to reviewing on a timely basis the first editions of the curriculum frameworks that were published in 1995. The purpose of this review process is to ensure that these statewide guidelines are useful to schools and districts and reflect accurate content.

This draft was written by the Mathematics Curriculum Framework Revision Panel, which is composed of teachers and administrators of mathematics programs in prekindergarten-12 school districts, university faculty, and mathematicians. The panel based its revisions on the first edition of the *Massachusetts Mathematics Curriculum Framework* (1995), NCTM's *Principles and Standards for School Mathematics* discussion draft (October 1998), feedback from the field, and current research in mathematics education.

This new mathematics draft presents:

- Clearer and more specific learning standards so that teachers at all levels will know what they are expected to teach. The skills and concepts are carefully designed to show a clear developmental sequence through the grades.
- Narrower grade spans. Within each strand, the standards are grouped by pairs of grades: PK-K, 1-2, 3-4, 5-6, 7-8, 9-10, and 11-12.
- A section showing which standards might be included in Algebra I, Geometry, and Advanced Algebra/Precalculus courses.

As the revision panel developed the learning standards at the high school level, the question was raised of whether to consider other options for the grade level and organization of the high school MCAS. Several options for MCAS assessment in high school emerged out of that discussion. The comment form in the front of this framework asks for your feedback on the options being considered.

Please take the time to review this revised curriculum framework carefully, looking both at your areas of specialization and at how the concepts and skills develop within and across grade ranges. We look forward to receiving your feedback on the contents of this framework; a comment and review form is included in this document. The panel has also requested that you send in problems and examples that can be used to effectively teach the standards. A selection of those submitted will be used in the final version of the framework. You can also participate in one of the discussion forums that the Department will be sponsoring throughout the public comment period. I hope that you will make copies available to others in your organization who are interested in mathematics education. To obtain additional copies, you can call the Department at (781) 388-3300 ext. 662 or access the framework draft at the Department's web site: www.doe.mass.edu.

Please send your comments by December 20, 1999, to:
Thomas Noonan
Massachusetts Department of Education
350 Main Street
Malden, Massachusetts 02148-5023
Fax: (781) 338-3395
E-mail: tnoonan@doe.mass.edu

After the panel has made its final revisions based on public comment, the Mathematics Framework will be presented to the Board of Education for formal acceptance.

Mathematics Curriculum Framework
Comment and Review Form

The panel earnestly seeks your feedback and will consider all comments as it revises and edits this draft. Your comments would be most helpful to the panel if you could:

- Cite the page and identify the specific standard(s) or line number in the draft OR
- Photocopy the page, mark your comments, and return it with this form.
- Identify specific concern(s) and offer a suggestion for improvement.
- If you have examples of problems or activities that effectively illustrate particular standards, please send them along with your comment form. Please include the author citation for appropriate acknowledgements.

Please mail or fax this response by December 20, 1999, to:

Thomas Noonan
Massachusetts Department of Education
350 Main Street
Malden, MA 02148-5023
Fax: (781) 338-3395
Email: tnoonan@doe.mass.edu

1. Name of person completing this form _____

School district or affiliation _____

Address _____
Street

_____
City/Town State ZIP code

Telephone _____ **Fax** _____

2. This review was completed by ___ **an individual** ____ **a group**

3. If by a group:
Description of group _____

Number of reviewers in group _____

4. Please identify the number of reviewer(s) in each category below.

Primary Role	**Teaching Role**	**Teaching Category**
___ Assessment Developer ___ Business Partner ___ Classroom Teacher ___ Curriculum Coordinator ___ Curriculum Developer ___ Department Chair ___ Higher Education Faculty ___ Library/Media Specialist ___ Parent ___ Principal ___ School Committee Member ___ Special Educator ___ Student ___ Superintendent ___ Assistant Superintendent ___ Other_____ ___ TOTAL (same as #3 above)	___ Grade PreK-4 Teacher ___ Grade 5-7 Teacher Indicate primary teaching area(s) _____ _____ Grade 8-12 Teacher Check primary teaching area(s) _____General Math _____Pre-Algebra _____Algebra _____Geometry _____Computer Science _____Trigonometry _____Probability/Statistics _____Pre-Calculus _____Calculus Other _____	___ Bilingual ___ Gifted and Talented ___ Regular Education ___ Special Education ___ Title I ___ Vocational Education Other _____

1.	**The guiding principles present a clear conceptual framework and rationale for learning, teaching, and assessing in PreK-12 programs of mathematics.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree Comments/Suggestions
2.	**Overall, the learning standards within each strand provide a strong foundation for mathematical literacy.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree Comments/Suggestions

3.	**Overall, the learning standards are specific enough to guide local curriculum and assessment development.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree Comments/Suggestions
4.	**In the future, the learning standards identified on page 45 for the Algebra I course should be incorporated into the grades 7-8 learning standards.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree Comments/Suggestions
5.	**The full set of learning standards identified on page 46 for a Geometry course comprise the core of a Geometry course taught by grade 10.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree Comments/Suggestions
6.	**The learning standards identified on page 47 for an Advanced Algebra/Precalculus course are _____ the core for an Algebra II course.** **Less than Equivalent to More than Cannot judge** Comments/Suggestions

7.	**The full set of learning standards identified on page 47 for an Advanced Algebra/Precalculus course adequately prepare students for a calculus course in grade 12 or a similarly advanced course.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree Comments/Suggestions
8.	**Number and Operations Strand** **The strand introduction gives a good rationale and overview of the strand.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree **The learning standards are clear and appropriate.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree **The exploratory concepts and skills are clear and appropriate.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree Comments/Suggestions:
9.	**Patterns, Functions, and Algebra Strand** **The strand introduction gives a good rationale and overview of the strand.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree **The learning standards are clear and appropriate.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree **The exploratory concepts and skills are clear and appropriate.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree Comments/Suggestions:

10.	**Geometry and Measurement Strand** **The strand introduction gives a good rationale and overview of the strand.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree **The learning standards are clear and appropriate.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree **The exploratory concepts and skills are clear and appropriate.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree Comments/Suggestions:
11.	**Data Analysis, Statistics, and Probability Strand** **The strand introduction gives a good rationale and overview of the strand.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree **The learning standards are clear and appropriate.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree **The exploratory concepts and skills are clear and appropriate.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree Comments/Suggestions:
12.	**Appendix I (Standards by Grade Span) is a useful chart.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree Comments/Suggestions

13.	**Appendix II (Instructional Technology) shows how to bring instructional technology into mathematics classrooms.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree Comments/Suggestions
14.	**Appendix V (Selected Problems and Activities) elucidates the intent of the standards.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree Comments/Suggestions
15.	**This curriculum framework provides sufficient guidance for classroom instruction.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree Comments/Suggestions
16.	**Should knowledge and use of standard algorithms in grades PreK-6 be explicitly expected in the learning standards and exploratory concepts and skills?** Yes No If yes, where?
17.	**Should knowledge and use of nonstandard algorithms be expected in the learning standards and exploratory concepts and skills?** Yes No If yes, where?

Assessment Issues

18.	**I prefer two options for the MCAS at the end of grade 8, one assessment based on all of the standards listed for grades 7 and 8 (as part of an integrated mathematics course), and another that is an end-of-course assessment for Algebra I.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree Comments/Suggestions
19.	**I prefer two options for the MCAS at the end of grade 10, one assessment based on all of the standards listed for grades 9 and 10 (as part of an integrated mathematics course), and another that is an end-of-course assessment for Geometry and/or the equivalent of an Algebra II course.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree Comments/Suggestions
20.	**I prefer a single assessment at the end of grades 8 and 10 based on the standards listed across all strands.** Strongly Agree Agree Cannot Judge Disagree Strongly Disagree Comments/Suggestions

If you have any other comments or suggestions about any part of this draft, please submit them on a separate sheet. Thank you for your input as we work toward improving mathematics education in Massachusetts.

Preface

The Massachusetts Mathematics Curriculum Framework

The Mathematics Curriculum Framework is one of seven such frameworks that advance Massachusetts' educational reform in learning, teaching, and assessment. It was created and has been revised by teachers and administrators of mathematics programs in prekindergarten through grade 12 school districts, university professors, and mathematicians working with staff from the Department of Education.

Organization of the Framework

The **guiding principles** provide the basis for a detailed set of underlying beliefs and tenets central to achieving in mathematics. These guiding principles articulate the ideals of teaching, learning, and assessing mathematics in Massachusetts.

The **strands and learning standards** section presents an outline upon which district and school curricula, instruction, and assessment can be developed. The mathematics framework is designed to be used in conjunction with the other six frameworks. The **strands** organize the content areas. They contain statements (called "learning standards") about what students should know and be able to do as learners of mathematics.

The **learning standards** within each strand are organized by grade span. The standards present key skills and concepts, and outline specifically what students should know and be able to do at the end of each grade span. Together with each set of learning standards are **exploratory concepts and skills** that are not assessed. They provide an introductory exposure to topics that will be more fully developed and assessed in subsequent learning standards.

This framework presents the learning standards that all Massachusetts students are expected to achieve in grades PreK-12. Following the PreK-12 standards are the fundamental standards for Algebra I , Geometry, and Advanced Algebra/Precalculus. Throughout this document, the standards are written to allow time for study of additional challenging material at every grade level and for advanced courses in grades 8 through 12. The high school standards are also written to allow for choice in course organization and sequence, for example, between a single-discipline course or an integrated approach. For schools choosing a single-discipline approach, the standards are written to allow the core for full-year courses in Algebra I, Geometry, and Advanced Algebra/Precalculus, in whatever order schools wish to teach these courses. All students are expected to pursue additional advanced study in mathematics such as advanced placement courses, independent research, internships, or study of special topics.

Developing the mathematics curriculum framework

The mathematics curriculum framework is based upon two reform initiatives in Massachusetts, the Education Reform Act of 1993 and Partnerships Advancing the Learning of Mathematics and Science (PALMS). PALMS is the Statewide Systemic Initiative, funded by the National Science Foundation since 1992. Of the seven initial goals for this initiative, the first was to develop, disseminate, and implement curriculum frameworks in mathematics and science and

technology. The initial mathematics framework was implemented in 1995, and the Education Reform Act required that it be reviewed and revised periodically. Since its release to districts and teachers, the framework has been read, critiqued, and implemented in the field. The Department of Education has received valuable feedback from educators, all of which has been taken into consideration as the framework has been revised.

The revision panel carefully examined the standards in the original framework, assessed their appropriateness individually, considered how they interfaced with the other strands, and worked to ensure the best progression through the grade levels. Since mathematics education reform is a national concern, the panel studied the reform efforts of national groups, noting in particular the work of the National Council of Teachers of mathematics. The panel also examined the research literature and reports of the Mathematics Association of America, the American Mathematical Society, and the American Association for the Advancement of Science. The panel used the *Principles and Standards for School Mathematics* discussion draft (October 1998) developed by the National Council of Teachers of Mathematics (NCTM); data from the Third International Mathematics and Science Study (TIMSS); the National Research Council's (NRC) *National Science Education Standards*; and results from the initial administration of the MCAS. The panel also reviewed National Science Foundation-funded curriculum projects and current textbook programs. Classroom experience and current educational research have also informed this revision. This framework is a work in progress. The Department of Education invites and encourages response from educators at all levels, parents, and partners in our community.

Massachusetts Mathematics Curriculum Framework
Visual Overview

Core Concept

Achieving in mathematics through problem solving, communicating, reasoning, and by making connections.

Guiding Principles

I.	Students explore mathematical ideas in ways that help them develop depth of understanding.
II.	An effective program focuses on the solving of problems and promotes the pursuit of further study of mathematics.
III.	Technology is an essential tool for effective mathematics education
IV.	All students should have access to mathematics curricula aligned with this framework.
V.	Mathematics assessment is a multifaceted process that monitors, enhances, and evaluates all students' learning and informs instruction.

Strands

Number and Operations	Patterns, Functions, and Algebra	Geometry and Measurement	Data Analysis, Statistics, and Probability

Core Concept

The vision of this curriculum framework is that all students in the Commonwealth achieve mathematical competence through a strong mathematics program that emphasizes problem solving, communicating, reasoning, and making connections.

Problem solving

Problem solving is both a powerful means of developing students' knowledge of mathematics and an indispensable outcome of a good mathematics education, and as such, it is an essential component of the curriculum. To become good problem solvers, students need many opportunities to formulate questions, model problem situations using a variety of means, generalize mathematical relationships, and solve problems in both mathematical and real-world contexts. To develop and apply strategies to a wide variety of problems and to interpret and evaluate results, a strong command of mathematics content is critical.

Communicating

The ability to express mathematical ideas coherently to different audiences is an important skill in an increasingly technological society. Students develop this skill and deepen their understanding of mathematics as they use accurate mathematical language to talk and write about what they are doing. They clarify mathematical ideas and definitions as they collaborate with other students, talk with and listen to experts, and reflect on and share ideas, strategies, and solutions. Reading in the content area of mathematics helps students understand and develop the skills of making convincing arguments and representing mathematical ideas verbally, pictorially, and symbolically. Reading what students write is an excellent way for teachers to identify students' understanding and misconceptions.

Reasoning

From the early grades, students develop their reasoning skills by making and testing mathematical conjectures, drawing logical conclusions, and justifying their thinking in age-appropriate ways. As they advance through the grades, students' arguments become more sophisticated and they are able to construct formal proofs. This process is supported by their experiences with following the reasoning of others, and validating and constructing logical arguments. With multiple opportunities to use patterns in mathematical problems, analyze mathematical situations, and deduce theorems, students learn to link conceptual and procedural knowledge.

Making Connections

Students develop a perspective of the field of mathematics as an integrated whole by understanding connections within and outside of the discipline. It is important for teachers to demonstrate the significance and relevance of the subject by exploring the connections that exist within mathematics, with other disciplines, and between mathematics and students' own experiences.

Mathematics Guiding Principles

Guiding Principle I
Students explore mathematical ideas in ways that help them develop depth of understanding.

Students learn mathematics when they understand it deeply and are able to use it effectively. To achieve mathematical understanding, students should be actively engaged in doing meaningful mathematics, discussing mathematical ideas, and applying mathematics in interesting and thought provoking situations. Students should have coherent and comprehensive curricula that foster mathematical development over time.

It is important to engage students in tasks designed to challenge them in multiple ways. Short- and long-term investigations that balance and connect procedures and skills with conceptual understanding, and enhance problem solving abilities are integral components of an effective mathematics program. Projects are excellent vehicles for involving students in exploration of mathematical concepts. Activities that build upon prior knowledge and provide opportunities to solve problems that become progressively deeper, broader, and more sophisticated help students achieve more advanced levels of proficiency. Mathematical tasks and experiences generate active discourse, promote the development of conjectures, and lead to an understanding of the necessity for mathematical proofs. Mathematics presented as an integrated whole leads to the development of other mathematical ideas, makes meaningful connections to areas both within and outside mathematics, and permits depth of study.

Teachers are a key factor in what mathematics students learn and how well they learn it. Therefore, it is important that teachers have the tools and support necessary to actively engage students in cognitively demanding and worthwhile tasks that reflect sound and significant mathematics.

Classroom climates should promote rich discourse where students are able to communicate, in a variety of ways, their level of understanding of mathematics content and processes. Students should have opportunities to formulate conjectures, prove or disprove them, and support their reasoning. Student understanding is developed through ongoing reflection about content and process.

Guiding Principle II
An effective program focuses on the solving of problems and promotes the pursuit of further study of mathematics.

An effective mathematics program clearly demonstrates that students achieve in mathematics through problem solving, communicating, reasoning, and making connections within mathematics and across disciplines. An effective mathematics program instills in all students a solid understanding of and respect for mathematics and its important role in today's society. An effective mathematics program provides coherence and continuity; it provokes student interest through the presentation of engaging tasks, activities, and projects such as those that relate to everyday living, interesting events, or careers.

Mathematical problem solving is the centerpiece of an effective mathematics program. It embraces a core of essential skills for many of today's careers. The rapid growth of information and technology has created tremendous job opportunities for those who have the requisite skills. The ability to reason and analyze quantitative issues is necessary to cope with an increasingly complex world.

Mathematical problem solving begins with a careful analysis of the problem itself and a recognition of the components of the problem. This action requires insight and practice with a spectrum of mathematical problems. In order to solve the smaller pieces of problems, the solver must possess a firm grasp of mathematical techniques and their underlying principles. Armed with this deeper knowledge, the student can use mathematics in a flexible way to attack various problems and devise different ways of solving any particular problem. This process nurtures student confidence and creates an incentive to learn new methods needed to solve other problems.

Mathematical problem solving encourages curiosity and calls for reflective thinking, persistence and hard work, learning from the ideas of others, and going back over one's own work with a critical eye. Success in solving mathematical problems helps to create a positive attitude toward and an abiding interest in mathematics.

Guiding Principle III
Technology is an essential tool for effective mathematics education.

All students at all levels need tools for learning mathematics. These tools may include measuring instruments, manipulatives, scientific and graphing calculators, and computers with appropriate software. Each of these tools, if properly used, can contribute to a rich learning environment for developing and applying mathematical concepts. Advanced technology tools, when integrated into a mathematics program, broaden the scope of mathematics to which students have access. The ever-increasing use of computers as communication devices and sources of information has an impact on schooling, including instruction in mathematics.

Manipulatives and advanced technology help students understand mathematical concepts. The use of these devices allows students to investigate situations of increasing complexity and contributes to the development of necessary mathematical intuition. Technology changes what mathematics is to be learned and when and how it is learned. Currently available technology provides a dynamic approach to mathematical concepts, such as functions, rates of change, and averages, that was not possible in the past. Some mathematics becomes more important because technology requires it, some becomes less important because technology replaces it, and some becomes possible because technology allows it.

The use of electronic devices to communicate ideas within the classroom and with mathematical resources outside of it (e.g., searching for information in external databases or consulting mathematics professionals) is an important benefit of greater availability of technology in schools. The use of the internet has become an important supplement to library resources. Technology can enhance a student's access to the curriculum, increase opportunities to interact with peers, and assist in other avenues of communication. Technology can be especially helpful in assisting students with special needs in regular and special classrooms, at home, and in the community.

Availability of fast electronic computational devices has changed the practice of mathematics in business, science and industry, and has had a significant impact on the nature of advanced mathematics research. Students must become familiar with the devices they are likely to be using at their jobs and at home upon completing their secondary education. Therefore, students whose education lacks a strong technology component may be disadvantaged when entering the workforce.

Guiding Principle IV
All students should have access to mathematics curricula aligned with this framework.

All Massachusetts students must have equal access to high quality mathematics programs that have clear goals and expectations for performance that match this framework. The framework addresses the educational requirements of the entire range of students, from those requiring tutorial support to those needing enrichment experiences. All students should be encouraged to achieve the highest level in mathematics and to develop facility with problem solving, reasoning, communicating, and making connections.

All students can learn mathematics through a coherent and comprehensive curriculum. This framework sets the standards for building such a curriculum. To promote achievement of these standards, classroom environments should encourage discourse, reflection, use of multiple problem-solving strategies, and positive dispositions toward mathematics. Teachers should expect that all students can grow mathematically.

All students should have equitable access to resources that enhance learning, including instructional materials, libraries, technology, and mentoring. Practice and enrichment should extend beyond the classroom. Tutorial sessions, mathematics clubs, mathematics competitions, apprenticeships, and community service projects are examples of extracurricular mathematics activities that promote learning.

Mathematics is the cornerstone of many disciplines. A robust curriculum should include context-based applications and modeling activities that demonstrate the connections among disciplines. Schools should provide opportunities for communicating with experts in applied fields to enhance students' knowledge of these connections. Communicating with experts also exposes students to mathematics and mathematics related careers.

Guiding Principle V
Mathematics assessment is a multifaceted process that monitors, enhances, and evaluates all students' learning, and informs instruction.

A comprehensive assessment program is aligned with the curriculum framework and is an integral component of the instructional program. It provides students with regular and frequent feedback on their performance and with opportunities for self evaluation; teachers with diagnostic tools for gauging students' depths of understanding of mathematical concepts and skills; parents with information about their children's performance relative to the goals of the program; and administrators with a means for measuring student achievement and comparing relative success of different student groups.

Assessment tools take a variety of forms, require varying amounts of time for their completion, and assess different aspects of student learning. One-to-one interviews provide teachers with opportunities to probe students' understanding of mathematical ideas and processes, and their abilities to reason mathematically. Having students "think aloud" or talk through their solutions to problems permits identification of gaps in knowledge and errors in reasoning. By observing students as they work on projects, teachers can gain insight into students' abilities to apply appropriate mathematical concepts and skills, make conjectures, draw conclusions, and collaborate with their peers. Homework, mathematics journals, and oral performances offer additional means for capturing students' thinking, knowledge of mathematics, facility with the language of mathematics, and ability to communicate what they know to others. Tests and quizzes assess knowledge of mathematical facts, operations, concepts, and skills and their efficient application to problem solving. Tests and quizzes can pinpoint areas in need of more practice or teaching. Taken together, the results of these assessments provide rich profiles of students' achievements in mathematics and serve as the basis for identifying curricula and instructional approaches to best develop their talents.

Assessment should be a major component of the learning process. As students help identify goals for lessons or investigations, particularly those that deal with mathematical content, communication, and the establishment of criteria for evaluating achievement of the goals, they gain greater awareness of what they need to learn and how they will demonstrate that learning. As students collaborate on projects, they learn to evaluate their own work and that of their peers. Engaging students in these kinds of assessment is a vital part of their abilities to reflect on their own work, understand the standards to which they are held accountable, and to take ownership of their learning.

Just as there are several forms of assessment, there are several levels at which assessment should be conducted. Classroom, district, and statewide (MCAS) assessment should be thoughtfully aligned over time to make the best use of resources and provide the richest array of data about student performance. Analysis of results will also help districts and teachers align their curricula with the framework standards and guide professional development.

Strand 1
Number and Operations

The study of numbers and operations is the cornerstone of the mathematics curriculum. Learning what numbers mean, how they may be represented, relationships among them, and computations with them are central to developing number sense and form the core of the Number and Operations strand. Beginning with prekindergarten and continuing through grade 12, students develop a robust understanding of the structure of the number system through exploration of whole numbers, integers, rational and irrational numbers, real numbers, and complex numbers.

Research in developmental psychology and in mathematics education has shown that young children have a great deal of informal knowledge of mathematics. As early as age three, children begin counting and quantifying, and demonstrate an eagerness to do so. Capitalizing on this informal knowledge and interest, education in the early years focuses on developing children's facility with oral counting and recognition of numerals and word names for numbers. Experience with counting naturally extends to quantification. Children count objects and learn that the sizes, shapes, positions, or purposes of objects do not affect the total number of objects in a group. One-to-one correspondence, with its matching of elements between two sets, provides the foundation for the comparison of groups. Combining and partitioning groups of objects sets the stage for operations with whole numbers, and the identification of equal parts of groups.

In the early elementary grades, students gain greater experience with counting and computing with whole numbers, learn different meanings of the operations and relationships among them, and apply the operations to the solutions of problems. As they progress through the grades, students learn to compute with multi-digit numbers, estimate to verify results of computations with larger numbers, and use concrete objects to model operations with fractions, mixed numbers, and decimals. By the end of their elementary school years, students should be able to choose operations appropriately, estimate to solve problems mentally, and compute with whole numbers.

The major foci of middle school mathematics are 1) computing with fractions, decimals and percents, and integers; and 2) the study of ratio and proportion (what they are and how they are used to solve problems). Students achieve competence with rational number computations and with the application of the order of operations rule in preparation for high school.

At the high school level, understanding systems of numbers is enhanced through formal exploration of real numbers and computations with them. Thereafter students investigate complex numbers and relationships between the real and complex numbers. Students expand their knowledge of counting techniques, permutations, and combinations, and apply those techniques to the solution of problems.

As students develop competence with numbers and computation, they construct the scaffolding necessary to build an understanding of number systems. Students not only learn to compute and solve problems with different types of numbers, but also explore the properties of operations on these numbers. Through investigation of relationships among sets of numbers (e.g., whole numbers are a subset of the integers, which are a subset of the rationals, which together with the irrational numbers constitute the set of real numbers), students gain a robust understanding of the structure of our number system.

Technology in the Number and Operations strand is used to facilitate investigation of mathematical concepts, skills, and strategies. Calculators and computers enhance students' abilities to explore relationships among different sets of numbers (e.g., the relationship between fractions and decimals, fractions and percents, and decimals and percents); investigate alternative computational methods (e.g., generating the product of a pair of multi-digit numbers on a calculator when the multiplication key cannot be used); verify results of computations done with other tools; compute with very large and very small numbers using numbers in scientific notation form; and learn the rule for the order of operations.

Number and Operations

All students will engage in problem solving, communicating, reasoning, and connecting as they:

Grades PreK – K

Learning Standards	Exploratory Concepts and Skills
1. Count by ones to at least 100. 2. Match quantities up to at least 20 with numerals. 3. Compare sets of up to at least 20 concrete objects using appropriate language (e.g., more than, fewer than, same number of, one more than), and order numbers. 4. Model and solve addition sums to 10 and subtraction situations using objects and drawings. 5. Identify positions of objects in sequences (e.g., first, second). 6. Estimate the number of objects in a group.	✓ Count by ones, beginning from any number in the counting sequence. ✓ Represent quantities using concrete objects, and investigate the partitioning of sets. Identify equal parts of groups. ✓ Create problems that can be solved using addition and subtraction.

The learning standards are numbered for reference only. The numbers do not indicate a priority order.

Number and Operations

Grades 1 – 2

Learning Standards	Exploratory Concepts and Skills
1. Identify and distinguish among multiple uses of numbers, including cardinal (to tell how many) and ordinal (to tell which one) numbers, and numbers as labels and as measurements. 2. Name and write (in numerals and words) whole numbers to at least thousands, identify the values of the digits, and order the numbers. 3. Use concrete materials to model addition and subtraction. 4. Describe various meanings of addition and subtraction, and the relationship between the two operations. 5. Know addition and subtraction facts (sums to 20), and use them to solve problems. 6. Find the value of a collection of coins less than $5.00 and different ways to represent an amount of money less than $5.00 with coins. 7. Identify common fractions (1/2, 1/3, 1/4) as parts of wholes, parts of groups, and numbers on a number line.	✓ Use concrete materials to investigate situations that lead to multiplication and division. ✓ Develop, explain, and use strategies for addition and subtraction of multi-digit whole numbers. Check by estimation. ✓ Investigate addition of common fractions, e.g., ½ + ½ = 1, ¼ + ¼ = ½

Number and Operations

Grades 3 – 4

Learning Standards	Exploratory Concepts and Skills
1. Name and write whole numbers to at least one million and decimals through hundredths, identify the values of the digits, and order the numbers. 2. Explain various meanings and models of multiplication and division of whole numbers and the inverse relationship between the two operations. 3. Know multiplication and division facts through 12 x 12 and use them to solve problems. 4. Apply commutative, associative, and identity properties of operations on whole numbers. 5. Add, subtract, multiply (up to three digits by two digits), and divide with a single-digit divisor (with remainders). 6. Apply a variety of mental mathematics and estimation methods to problems involving up to three-digit whole numbers and amounts of money to $1000. 7. Use models to relate fractions (1/2, 1/3, 1/4, 1/5, 1/6, 1/8, 1/10, and 1/12) to decimals, find equivalent fractions, and order fractions. 8. Use concrete objects to model operations with fractions and decimals. 9. Choose the appropriate operations (addition, subtraction, multiplication, division) to solve problems, including those involving money, and solve the problems.	✓ Investigate and develop procedures for multi-digit multiplication (up to 3-digit by 2-digit), and division with a single-digit divisor with remainders. ✓ Use models to explore operations with fractions (to twelfths) and decimals (including amounts of money). ✓ Investigate number theory concepts, e.g., odd/even, factors, multiples, prime, and composite numbers. ✓ Investigate ratio and proportion.

Number and Operations

Grades 5 – 6

Learning Standards	Exploratory Concepts and Skills
1. Represent very large (billions, trillions, …) and very small (thousandths, millionths, …) positive numbers in various forms, including expanded notation with exponents. 2. Apply ratios and proportions to the solution of problems. 3. Describe relationships among fractions, decimals, and percents. 4. Solve problems involving addition, subtraction, multiplication, and division with whole numbers, fractions, and decimals (including percents). 5. Estimate results of computations with whole numbers, fractions, decimals, and percents. 6. Compare and order whole numbers, decimals, and fractions. 7. Apply the Order of Operations, including parentheses. 8. Apply number theory concepts, including prime and composite numbers, prime factorization, greatest common factor, least common multiple, and divisibility to the solution of problems.	✓ Use models to explore integers and computations with integers. ✓ Compare and order whole numbers, decimals, fractions, and integers.

Number and Operations

Grades 7 – 8

Learning Standards	Exploratory Concepts and Skills
1. Estimate and compute with fractions, decimals, percents, and integers. 2. Represent numbers in scientific notation, and use them in problem situations. 3. Apply powers and roots to the solution of problems. Extend the Order of Operations to include exponents. 4. Define and apply frequently used irrational numbers, e.g., $\sqrt{2}$, π. 5. Describe conditions under which an estimate rather than an exact answer is appropriate. Apply in problem situations. 6. Apply number theory concepts, including relatively prime numbers, to the solution of problems. 7. Apply the Fundamental Counting Principle to the solution of problems. 8. Identify the properties of operations on integers and rational numbers, including closure, associativity, commutativity, distributivity, identity, and inverse.	✓ Investigate the meaning of significant digits.

Number and Operations

Grades 9 – 10

Learning Standards	Exploratory Concepts and Skills
1. Apply operations with powers and roots, including fractional and negative exponents, to the solution of problems. 2. Use estimation to judge the reasonableness of results of computations and of solutions to problems involving real numbers.	✓ Analyze relationships among the various subsets of the real numbers (whole numbers, integers, rationals, and irrationals).

Grades 11 – 12

Learning Standards	Exploratory Concepts and Skills
1. Describe the structure and properties of the real number system and relationships between the real number system and its various subsets. 2. Define complex numbers and operate with them. 3. Represent finite graphs using matrices and apply them to the solution of problems. 4. Use combinatorics (e.g., permutations and combinations) to solve problems.	✓ Investigate special topics in number theory, e.g., number families.

Strand 2
Patterns, Functions, and Algebra

Patterns, functions, and algebra are integral to the study of mathematics. All students must be provided with a rich instructional program that examines a broad range of patterns, builds a solid understanding of the concept of function, and ensures that students are fluent in using verbal and symbolic language and appropriate technology to represent relationships and to do mathematical modeling.

Readiness for the study of patterns, functions, and algebra should begin in the PreK-K years of school. A strong foundation should be carefully constructed and expand gradually as the years of schooling progress. All students should be comfortable with basic algebraic concepts and the meaning of function by the end of eighth grade. The treatment of algebra and functions should accelerate and continue to grow throughout the secondary years.

All students should be aware of the mathematics in patterns and use mathematical representations to describe patterns. Young students identify, translate, and extend repeating rhythmic, verbal, and visual patterns. They learn to recognize patterns and relationships among objects, and to sort and classify them, observing similarities and differences. Young students then probe more deeply into the study of patterns as they explore the properties of the operations of addition and multiplication.

Through numerous explorations, elementary grade students deepen their own understanding of pattern and work informally with the concept of function. It is important that the concept of a variable is developed for them through practical situations, for example, as they engage in such basic activities as listing the cost of one pencil at 50¢, two pencils at ?, three pencils at ? , ... n pencils at ?

Investigating patterns helps older students understand the concept of constant growth as they analyze sequences like 1, 3, 5, 7, ... These students should be given opportunities to contrast this type of change with other relationships as evidenced in sequences such as 1, 3, 6, 10, 15, ... ; 1, 2, 4, 8, ... ; and 1, 1/10, 1/100, 1/1000, ... In middle and high school, students build on prior experiences and formalize recursive thinking as they compare sequences and functions represented in recursive and in closed forms.

Algebra emerged through the analysis of solutions to equations while the concept of a function developed as the roots of calculus began to spread. Currently, school level algebra coursework includes abstract reasoning, methods for solving equations and inequalities, and the study of functions and mathematical structures. Students need to know that algebra is a branch of mathematics that uses symbolic notation to express ideas, make generalizations, and solve problems. Elementary and middle school students should use concrete models to build their understanding of algebraic manipulations and make the connection with symbolic notation. They should realize that the language of algebra can be viewed as a form of shorthand that facilitates problem solving. Secondary

students should become comfortable using the language of algebra to construct mathematical models and be successful in solving a variety of applied problems.

As students matriculate through school, their work with patterns, functions, and algebra progresses in mathematical sophistication. They learn that change is a central idea in the study of mathematics and that multiple representations are needed to express change. They identify, represent, and analyze numerical relationships in tables, charts, and graphs. They learn about the importance and power of proportional reasoning as a means of solving a variety of problems. While understanding linear functions and their graphs is a realistic goal for the middle school student, students should deepen their study of functions in the secondary years. They should engage in problems that feature additional types of polynomial functions as well as rational, exponential, logarithmic, trigonometric, and other families of functions. Graphing calculators and computer software with spreadsheet and graphics capabilities are ideal resources to use as students explore and learn about functions. The meaning of domain, range, roots, optimum values, periodicity, and other terms come alive when experienced through technology. With appropriate instruction, students should be able to move readily among symbolic, numeric, and graphic representations of functions. Through insightful examples, secondary students learn that function is a key concept with connections not only to calculus but also to transformational geometry and topics in discrete mathematics.

Patterns, Functions, and Algebra

All students will engage in problem solving, communicating, reasoning, and connecting as they:

Grades PreK – K

Learning Standards	Exploratory Concepts and Skills
1. Sort and classify objects by color, shape, size, and number. 2. Reproduce, describe, extend, and create color, shape, number, and letter repeating patterns.	✓ Investigate growing and shrinking patterns.

Grades 1 – 2

Learning Standards	Exploratory Concepts and Skills
1. Reproduce, describe, extend, and create simple rhythmic, shape, size, number, and color repeating patterns. 2. Describe and create addition and subtraction number patterns. 3. Write number sentences to represent mathematical relationships in real-world situations. 4. Construct and solve open sentences that have variables, e.g., □ + 7 = 10, □ + ❍ = 10. 5. Solve problems with input-output function machines and tables of data. 6. Describe functions related to trading, including coin trades (e.g., pennies for nickels, nickels for dimes) and measurement trades (e.g., cups for quarts).	✓ Investigate situations with variables as unknowns and as quantities that vary.

Patterns, Functions, and Algebra

Grades 3 – 4

Learning Standards	Exploratory Concepts and Skills
1. Describe, extend, and create geometric and numeric patterns; make predictions and form generalizations about these patterns. 2. Identify and describe patterns created by multiples of whole numbers. 3. Use variables to represent unknowns in quantities that vary in expressions and in equations (mathematical sentences). 4. Describe procedures for finding values of variables in equations. 5. Solve problems involving proportional relationships, including unit pricing (e.g., four apples cost 80¢, so one apple costs 20¢) and map interpretation (e.g., one inch represents five miles, so two inches represents ten miles).	✓ Use concrete materials to build an understanding of equality and inequality, and ways to maintain these relations.

Grades 5 – 6

Learning Standards	Exploratory Concepts and Skills
1. Extend and generalize numeric and geometric patterns. 2. Describe strategies for solving linear equations using concrete models, tables, graphs, and paper-pencil methods. 3. Produce and interpret graphical sketches that represent real events. 4. Represent real situations and mathematical relationships with concrete models, tables, graphs, and rules in words and with symbols. 5. Model situations with proportional relationships and solve problems.	✓ Develop conceptual understanding of *equation* and *variable*. ✓ Use physical models to investigate and describe how a change in one variable affects a second variable. ✓ Use models to develop understanding of slope as constant rate of change.

Patterns, Functions, and Algebra

Grades 7 – 8

Learning Standards	Exploratory Concepts and Skills
1. Describe, complete, extend, analyze, generalize, and create a wide variety of patterns, including iterative and recursive (e.g., Pascal's triangle), and linear and nonlinear functional relationships. 2. Use tables and graphs to compare linear, quadratic, and exponential growth patterns. 3. Demonstrate proficiency in solving linear equations and their applications using technology, graphs, and algebraic methods. 4. Represent and solve single-variable inequalities using symbols and graphs. 5. Identify the slope of a line as a constant rate of change from its table of values, equation, and graph. Apply the concept of slope to the solution of problems. 6. Explain and generalize how a change in one variable results in a change in another variable in functional relationships, e.g., $C = \pi d$, $A = \pi r2$, $A_{rectangle} = lw$, and $A_{triangle} = (2\ )bh$.	✓ Investigate the use of systems of equations, tables, and graphs to represent mathematical relationships.

Patterns, Functions, and Algebra

Grades 9 – 10

Learning Standards	Exploratory Concepts and Skills
1. Demonstrate facility in transforming polynomial expressions by rearranging and collecting terms, factoring, and applying the properties of exponents in order to solve problems. 2. Identify problem situations that lead to linear, quadratic, or exponential equations and solve by applying appropriate graphical, tabular, or symbolic methods. Describe relationships among the methods. 3. Use algebraic and graphical methods to solve systems of linear equations and inequalities and describe relationships between different solution methods. 4. Use systems of equations or inequalities to represent mathematical relationships and to solve problems. 5. Describe similarities and differences among the families of linear, quadratic, and exponential functions using graphs, tables, formulas, and verbal descriptions. Describe the graphical significance of parameters. 6. Solve problems involving direct and inverse variation.	✓ Explore matrices and their operations. Use matrices to represent situations with variable quantities and to solve systems of linear equations. ✓ Investigate recursive function notation.

Patterns, Functions, and Algebra

Grades 11 – 12

Learning Standards	Exploratory Concepts and Skills
1. Describe and model phenomena using functions, including exponential, logarithmic, trigonometric, polynomial, rational, step, absolute value, and square root. 2. Solve polynomial, exponential, logarithmic, and trigonometric equations and equations with rational expressions by symbolic (quadratic), graphical, and numerical methods. Apply each method when appropriate. 3. Solve systems of equations and inequalities involving algebraic, exponential, logarithmic, and trigonometric expressions using symbolic, numeric, and graphical methods. Describe the relationships among the methods. 4. Classify functions into families. Describe the effects of parameter changes on different representations of polynomial, rational, logarithmic, exponential, and trigonometric functions. 5. Identify maximum and minimum values of functions and use them in applications. 6. Perform operations on functions, including compositions. Find inverses of functions. 7. Model real-world phenomena involving growth, decay, and periodic processes. 8. Identify arithmetic and geometric sequences and series and their properties. Solve problems, including finding the n^{th} term recursively and explicitly. 9. Define linear, exponential, and quadratic functions recursively and to find the closed form expressions.	✓ Investigate parametric functions and recursively defined functions, including applications to dynamical systems. ✓ Investigate the binomial theorem.

Strand 3
Geometry and Measurement

Geometry, spatial sense, and measurement were the source of the earliest mathematical endeavors. Ancient people, in their efforts to understand their surroundings and to describe the beauty and regularity of the natural forms, resorted to abstract shapes and standard units of measurements to communicate with each other. Today, students are confronted with many of the same needs their ancestors confronted thousands of years ago. They need to understand the structure of space and the spatial relations around them, measure many aspects of their environment, and communicate this structure and these relations and measurements to others. To address these needs, all students should be given opportunities to develop their spatial sense and measurement abilities.

Before students begin school, they have developed some knowledge of and a natural curiosity about the physical and spatial world. They explore size, shape, position, and orientation of objects when they are involved in everyday activities at home, at school, in the neighborhood playground, or at a supermarket. They become familiar with two- and three-dimensional shapes and develop a need to classify and categorize them. The language describing location and orientation of objects—words such as right, left, above, below, top, bottom, and between—play an increasingly important role in children's communicative processes. Children use the language of measurement and geometry in their everyday communication. However, teachers must be aware that the *everyday* use of the language of measurement and geometry often differs from the *formal* use in mathematics, and that these differences must be addressed through instruction. At all grade levels, students should be encouraged to use precise language, previously discovered relationships, and appropriate explanations for their conclusions and conjectures.

In the early grades, children should have opportunities to explore shapes and the relationships among them that build on their natural and intuitive understanding. As they progress through elementary school, students identify the components, attributes, and properties of different shapes, including sides, corners or vertices, edges, interiors, and exteriors. With time, students develop procedures to identify and categorize shapes by referring to their components, attributes, and properties. Students should have opportunities to investigate these features dynamically, by using mirrors, paper folding, paper-and-pencil, and computer drawing. Still operating on concrete objects, students can develop the idea of transformations by recognizing changes affected by slides, flips, and turns, not only on individual objects, but on combinations of objects. Investigations of simple transformations lead to the concept of congruence.

In middle school and high school, students solve problems from other areas of mathematics by interpreting them geometrically, and use alternative representations, including coordinate geometry, perspective drawings, and projections of three-dimensional objects. Mechanical and electronic tools are used to perform constructions of common geometric shapes and patterns, and to develop the idea of geometric similarity, which can be integrated with the ideas of ratio and proportion.

Students' use of formal reasoning and proof should increase as they progress through school. However, this increased formalization of reasoning should not obscure the value of geometric ideas as problem-solving tools in mathematical and non-mathematical contexts. Students can apply methods developed in the geometric context to make sense of fractions and variables, construct graphs and other representations of data, and to make and interpret maps, blueprints, and schematic drawings.

In high school, students use formal reasoning to justify conclusions about geometry. They should recognize the logical structure of the system of geometric axioms, become increasingly proficient in proving theorems within the axiomatic system, and use axioms and theorems to verify conjectures generated through their own work or by their peers. Students learn to apply coordinate geometry to the solution of problems. They extend their experience with transformational geometry to a variety of congruence and similarity transformations and their composition.

Measurement, like no other aspect of science, is responsible for the historical development of mathematics. It is one topic that naturally lends itself to connections within mathematics and across other disciplines, including the social, physical, and biological sciences. Teaching measurement is best accomplished through direct applications or by being embedded within other mathematical topics.

A measurable attribute of an object is a characteristic that is most readily quantified and compared. Many attributes, such as length, perimeter, area, volume, and angle measure come from the geometric realm. Other attributes are physical, such as temperature and mass. Still other attributes are not readily measurable by direct means, as for example, speed and density.

Relying on student-constructed informal units of measurement, students in the early grades should be encouraged to make quantitative comparisons between physical objects, e.g., which object is longer or shorter? which is lighter or heavier? which is warmer or colder? Building on existing informal ideas, students gain appreciation for and become competent with standard units of measurement. Throughout their study of measurement techniques, students become familiar and facile with appropriate measurement tools.

As students gain understanding of ratio and proportion in the middle grades, they can apply their newly found knowledge to making scale drawings and maps that accurately reflect the dimensions of the landscape or the objects they represent. Greater familiarity with ratios enhances students' understanding of the derived attributes (speed, density, trigonometric ratios), their applications, and the use of conversion factors to change a base unit in a measure.

At all levels, students must be aware of the issues of precision and accuracy in measurement. As they grow in mathematical sophistication, students learn to select the tools and the units of measurement appropriate to the situation. With time, they learn to analyze the possible and the certain errors of their applied measurements and the results of compounding those errors by using measurements in computations.

Geometry

All students will engage in problem solving, communicating, reasoning, and connecting as they:

Grades PreK – K

Learning Standards	Exploratory Concepts and Skills
1. Name, describe, and draw simple two-dimensional shapes, and identify shapes that have been rotated, reflected, or enlarged. 2. Describe attributes of two-dimensional shapes, e.g., number of sides, number of corners. 3. Name and describe three-dimensional concrete objects; compare attributes, e.g., number of faces, shape of faces. 4. Identify positions of objects in space, and use appropriate language (e.g., beside, inside, next to, above, below, nearer, farther) to describe and compare their relative positions.	✓ Investigate symmetry of two- and three-dimensional shapes and constructions.

Grades 1 – 2

Learning Standards	Exploratory Concepts and Skills
1. Identify, describe, draw, and compare two-dimensional shapes, including both polygonal and curved figures. 2. Describe attributes and parts of two- and three-dimensional shapes, e.g., number and length of sides, number of corners, edges, and faces. 3. Predict the results of putting shapes together and taking them apart. 4. Identify symmetry in two-dimensional shapes. 5. Recognize congruent shapes.	✓ Investigate symmetry in two-dimensional shapes with mirrors or by paper folding. ✓ Explore intersecting, parallel, and perpendicular lines. ✓ Develop map reading skills.

Geometry

Grades 3 – 4

Learning Standards	Exploratory Concepts and Skills
1. Compare and analyze attributes of two- and three-dimensional geometric shapes, including diagonals. 2. Define and differentiate among various quadrilaterals, including squares, rectangles, rhombuses, parallelograms, and trapezoids. 3. Predict and validate the results of combining and partitioning two- and three-dimensional shapes. 4. Identify lines of symmetry in two-dimensional shapes. 5. Graph points in the first quadrant of the coordinate plane and identify the coordinates of points. Make and use coordinate maps to represent actual places. 6. Describe and draw intersecting, parallel, and perpendicular lines.	✓ Describe effects of transformations (e.g., rotations and reflections) on shapes. ✓ Investigate two-dimensional representations of three-dimensional objects.

Grades 5 – 6

Learning Standards	Exploratory Concepts and Skills
1. Identify properties of polygons. 2. Describe the relationship among points, lines, and planes. 3. Match three-dimensional objects and their two-dimensional representations (e.g., nets, projections, and perspective drawings). 4. Graph points on the Cartesian coordinate plane and identify coordinates of points. 5. Describe and perform transformations on shapes, e.g., translations, rotations, and reflections. 6. Describe and apply techniques for determining if two shapes are congruent.	✓ Use manipulatives and technology to model geometric shapes. ✓ Investigate tessellations (tilings). ✓ Explore the angles formed by intersecting lines.

Geometry

Grades 7 – 8

Learning Standards	Exploratory Concepts and Skills
1. Explain the meaning of the Pythagorean Theorem, and apply the theorem to the solution of problems. 2. Describe angles formed by intersecting lines and relationships among them. 3. Predict the results of transformations of the coordinate plane and draw the transformed figure. 4. Use technology, straightedge, compass, or other tools to formulate and test conjectures and to make geometric constructions. 5. Classify figures in terms of congruence and similarity, and apply these relationships to the solution of problems. 6. Apply networks to the solution of problems.	✓ Formulate and test conjectures about shapes that tessellate. ✓ Investigate trigonometric ratios in right triangles. ✓ Investigate right triangle relationships.

Geometry

Grades 9 – 10

Learning Standards	Exploratory Concepts and Skills
1. Use deduction to establish the validity of geometric conjectures and to prove theorems in Euclidean geometry. 2. Construct congruent and similar figures using a compass, straightedge, manipulatives, and other tools. 3. Derive and apply properties of angles, arcs, chords, tangents, and secants to solve problems involving circles. 4. Apply the geometry of linear graphs to parallel and perpendicular lines, and perform and interpret transformations of coordinates. 5. Apply trigonometric ratios in right triangles to solve problems. 6. Apply and interpret transformations on figures in the coordinate plane, e.g., translations, reflections, rotations, scale factors or size changes, and the results of successive transformations. 7. Describe characteristics of discrete geometry (graph theory) and apply to the solution of problems. 8. Identify and describe geometric patterns of change using recursive notation.	✓ Incorporate the use of technology such as CBL, CBR, software, and graphing calculators to investigate the graphical representation of functions and the properties of transformations. ✓ Investigate vector geometry.

Geometry

Grades 11 – 12

Learning Standards	Exploratory Concepts and Skills
1. Derive and apply trigonometric identities and the laws of sines and cosines. 2. Use transformations and coordinate geometry to represent real-world and mathematical situations. 3. Use vector geometry to solve problems. Describe addition of vectors and scalar multiplication both symbolically and pictorially. Use vector methods to obtain geometric results. 4. Relate geometric and algebraic representations of curves, including conic sections.	✓ Investigate and compare the axiomatic structures of Euclidean and non-Euclidean geometries. ✓ Explore the use of conic sections in engineering, design, and other applications. ✓ Investigate patterns of self-similarity and relate them to iterations and sequences.

Measurement

All students will engage in problem solving, communicating, reasoning, and connecting as they:

Grades PreK – K

Learning Standards	Exploratory Concepts and Skills
1. Use nonstandard and standard units to measure length, area, weight, and capacity. 2. Compare lengths, weights, capacities, and temperature using appropriate language, e.g., longer, taller, shorter, same length; heavier, lighter, same weight; holds more, holds less, holds the same amount; warmer, cooler, same temperature. 3. Identify hour and half-hour times using analog and digital clocks.	✓ Explore and use standard units to measure and compare temperature, length, and time. ✓ Identify coins and their values. ✓ Identify positions of events over time (e.g., earlier, later).

Grades 1 – 2

Learning Standards	Exploratory Concepts and Skills
1. Identify parts of the day (e.g., morning, afternoon, evening), week, month, and calendar. 2. Tell time at five-minute intervals on analog and digital clocks. 3. Identify pennies, nickels, dimes, quarters, $1, $5, $10, and $20 bills, and compute values of collections of coins and bills. 4. Make and use estimates of measurement, including time. 5. Measure common objects using metric and customary units of length and weight measurement, e.g., centimeter, inch, gram, and ounce; compare and order objects by some measure. 6. Select and use appropriate measurement tools, e.g., ruler, balance scale, and thermometer.	✓ Explore measures of objects, including length, perimeter, weight, area, volume, temperature, and angle measure. Compare concrete objects using these measures.

Measurement

Grades 3 – 4

Learning Standards	Exploratory Concepts and Skills
1. Identify time to the minute on analog and digital clocks. 2. Compute elapsed time, and make and interpret schedules. 3. Identify and use linear and square units of measurement appropriate for a particular situation. 4. Apply the formula for the area of a rectangle. 5. Identify angles as acute, right, or obtuse.	✓ Develop the concept of area further by investigating areas of regular and irregular shapes drawn on coordinate grids or dot paper, or created on geoboards. ✓ Use concrete objects to explore volumes of rectangular prisms, and construct rectangular prisms with different dimensions that have equal volumes. ✓ Investigate the use of protractors and angle rulers to measure angles.

Grades 5 – 6

Learning Standards	Exploratory Concepts and Skills
1. Solve problems involving proportional relationships and units of measurement, e.g., scale models, maps, and speed. 2. Find areas of triangles and parallelograms. Recognize that skewed versions of the same shape have the same area. 3. Apply the concept of perimeter to the solution of problems, and contrast it with the concept of area. 4. Identify, measure, describe, classify, and construct various angles and triangles. 5. Find sums of angle measures of polygons. 6. Describe the relationships of the radius, diameter, circumference, and area of a circle (e.g., $d = 2r$, $c/d = \pi$). 7. Find volumes of rectangular prisms.	✓ Explore various models for finding the area of a triangle. ✓ Investigate volumes and surface areas of three-dimensional objects.

Measurement

Grades 7 – 8

Learning Standards	Exploratory Concepts and Skills
1. Generalize the relationships between the number of sides and the sums of the angle measures of polygons. 2. Apply formulas and procedures for determining measures, including areas of polygons, and volumes and surface areas of prisms, cylinders, and spheres. 3. Use proportions to model and solve indirect measurement problems. 4. Identify proportional relationships in similar plane figures and apply to the solution of problems. 5. Select and use an appropriate unit of measurement or scale.	✓ Investigate ratios of similarity.

Grades 9 – 10

Learning Standards	Exploratory Concepts and Skills
1. Relate changes in the measurement of one attribute of an object to changes in other attributes (e.g., how changing the radius or height of a cylinder affects its surface area or volume). 2. Use simple tools (such as a clinometer) to solve indirect measurement problems. 3. Apply the ratio of similarity, including the geometric mean and the relationships of special triangles, to the solution of problems. 4. Describe the effects of rounding on measurements and on computed values from measurements. 5. Apply formulas for surface area and volume of pyramids and cones.	✓ Explore scientific use of different systems of measurement, e.g., CGS, SI.

Measurement

Grades 11 – 12

Learning Standards	Exploratory Concepts and Skills
1. Describe the relationship between degree and radian measures, and use radian measure in the solution of problems, e.g., angular velocity and acceleration. 2. Use dimensional analysis for unit conversion and to confirm that expressions and equations make sense.	

Strand 4
Data Analysis, Statistics, and Probability

Education in a democratic society must prepare citizens to make informed choices in their careers, their individual and family lives, and their government. They must be able to grasp the information being presented, analyze it, and make reasoned decisions. To accomplish these goals, students in the early grades learn how to collect data, observe patterns in the data, and organize and analyze the data to draw conclusions. To organize and display their data, they begin by using concrete and pictorial representations, and gradually learn to use tables, bar charts, and line graphs. As students advance through the grades, they explore more complex forms of representation, including multiple-line graphs, circle graphs, and frequency tables.

In their study of data and statistics, students shift their perspective from viewing data as a set of individual pieces to an understanding of data as a coherent set with its own collective properties. This shift is emphasized in the middle grades when students study characteristics of sets of data, including measures of central tendency, and techniques for displaying these characteristics (e.g., stem-and-leaf and box-and-whisker plots). Students learn how to select and construct representations most appropriate for the data and how to avoid misleading and inappropriate representations.

In high school, students gain insight into the use of trend lines and measures of spread for analyzing data. Students use technology to find lines of best fit. They categorize data by the type of model that best represents them (e.g., linear, quadratic, exponential). They design surveys to generate data. In the design process, they learn to choose representative samples and identify biases in the samples and survey questions.

Probability may be called the study of the laws of chance. In the elementary grades, students begin the study of probability by conducting experiments with spinners, counters, number cubes, and other concrete objects. They learn to record outcomes of individual experiments, and to organize and analyze results. They identify certain, possible, and impossible events.

In the middle grades, students enumerate all possible outcomes of simple experiments and determine probabilities to solve problems. Through the exploration of various problem situations, students learn to distinguish between independent and dependent events.

In high school, as they compare results of experiments with their theoretical predictions, students gain understanding of the difference between the theoretical and experimental probabilities of an event occurring in experiments and simulations. They apply counting techniques to solve more complex probability problems, and they investigate probability distributions, including uniform, normal, and binomial distributions.

Data Analysis, Statistics, and Probability

All students will engage in problem solving, communicating, reasoning, and connecting as they:

Grades PreK – K

Learning Standards	Exploratory Concepts and Skills
1. Represent data using concrete models.	✓ Collect and organize data in lists, tables, and simple graphs.

Grades 1 – 2

Learning Standards	Exploratory Concepts and Skills
1. Use interviews, surveys, and observations to gather data about themselves and their surroundings. 2. Organize, classify, and represent data using tallies, charts, tables, bar graphs, pictographs, and Venn diagrams; interpret the representations. 3. Formulate inferences (draw conclusions) and make hypotheses (educated guesses). 4. Decide which outcomes of experiments are most likely. 5. List and count the number of possible pairings of objects from two sets, e.g., how many different outfits can one make from a set of three shirts and a set of two skirts?	✓ Investigate more likely, likely, and impossible outcomes by conducting experiments using spinners, counters, and other concrete objects.

Grades 3 – 4

Learning Standards	Exploratory Concepts and Skills
1. Construct and interpret various representations of data sets, including tables, bar graphs, line graphs, and pictographs. 2. Classify outcomes as likely, unlikely, certain, and impossible by designing and conducting experiments using concrete objects, counters, number cubes, or coins.	✓ Pose survey questions and collect and interpret data in response to these questions. ✓ Generate and group data. Record the data using frequency tables and interpret the tables. ✓ Explore the concepts of median, mean, mode, maximum and minimum, and range.

Data Analysis, Statistics, and Probability

Grades 5 – 6

Learning Standards	Exploratory Concepts and Skills
1. Describe data sets using the concepts of median, mean, mode, maximum and minimum, and range. 2. Use tree diagrams and other models (e.g., lists and tables) to represent outcomes of experiments. Analyze the outcomes. 3. Construct and interpret stem-and-leaf plots and line plots. 4. Find probabilities of events with equally likely outcomes.	✓ Explore situations that involve probabilities of equally likely events. ✓ Investigate the use of circle graphs. ✓ Set up and analyze capture-recapture experiments.

Grades 7 – 8

Learning Standards	Exploratory Concepts and Skills
1. Choose and apply appropriate measures of central tendency (mean, median, and mode) to represent a set of data. 2. Use tree diagrams, tables, and lists to describe sample spaces and to calculate probabilities of independent and dependent events. 3. Differentiate between continuous and discrete data and ways to represent them. 4. Make inferences about a characteristic of a population from a well constructed sample, e.g., capture-recapture. 5. Construct and interpret circle graphs. 6. Use box-and-whisker plots to represent data sets, and identify outliers.	✓ Investigate mathematical fairness in games.

Data Analysis, Statistics, and Probability

Grades 9 – 10

Learning Standards	Exploratory Concepts and Skills
1. Represent data in a scatterplot. Use the scatterplot to make predictions. 2. Find a line of best fit from a set of data. 3. Select an appropriate graphical representation for a set of data and use appropriate statistics (e.g., mean, median, range, quartile, or percentile distribution) to communicate information about the data. 4. Describe the effect of sample size and population size on the validity of predictions from a set of data. 5. Design a fair game and provide a justification for fairness. 6. Describe the differences between the theoretical probability of simple events and the experimental probability from simulations. 7. Apply basic counting principles to describe simple events, and compute probabilities of events with outcomes that are not equally likely.	✓ Explore designs of surveys, polls, and experiments to assess the validity of their results and to identify potential sources of bias; identify the types of conclusions that can be drawn.

Data Analysis, Statistics, and Probability

Grades 11 – 12

Learning Standards	Exploratory Concepts and Skills
1. Describe a set of frequency distribution data by spread, skewness, symmetry, number of modes, or other characteristics. 2. Apply regression results and curve fitting to make predictions from data. 3. Use measures of spread of a set of data (variance, standard deviation) to solve problems. 4. Design surveys and apply random sampling techniques to avoid bias in the data collected. 5. Use simulations (e.g., random number tables, random functions, and area models) to determine experimental probabilities. 6. Apply uniform, normal, and binomial distributions to the solutions of problems.	✓ Use graphs (networks) and matrices to investigate probabilistic processes, including Markov Chains. ✓ Use technology to perform linear, quadratic, and exponential regression on a set of data. ✓ Use technology to explore simulations for determining experimental probabilities. ✓ Explore the application of the principle of mathematical induction to the solution of problems.

Learning Standards by Course at the Secondary Level

On the following pages, the learning standards are organized by course for those interested in assessing learning in Algebra I, Geometry, and Advanced Algebra/Precalculus.

Algebra I

The developmental sequence requires that students must know and be able to do all of the concepts and skills in grades 1 – 8, plus the following grade 9-10 standards:

Number and Operations

- Apply operations with powers and roots, including fractional and negative exponents, to the solution of problems.
- Use estimation to judge the reasonableness of results of computations and of solutions to problems involving real numbers.

Patterns, Functions, and Algebra

- Demonstrate facility in transforming polynomial expressions by rearranging and collecting terms, factoring, and applying the properties of exponents to the solution of problems.
- Identify problem situations that lead to linear, quadratic, or exponential equations and solve by applying appropriate graphical, tabular, or symbolic methods. Describe relationships among the methods.
- Use algebraic and graphical methods to solve systems of linear equations and inequalities and describe relationships between different solution methods.
- Use systems of equations or inequalities to represent mathematical relationships and to solve problems.
- Describe similarities and differences among the families of linear, quadratic, and exponential functions using graphs, tables, formulas, and verbal descriptions. Describe the graphical significance of parameters.
- Solve problems involving direct and inverse variation.

Data Analysis, Statistics, and Probability

- Represent data in a scatterplot. Use the scatterplot to make predictions.
- Find a line of best fit from a set of data.
- Select an appropriate graphical representation for a set of data and use appropriate statistics (e.g., mean, median, range, quartile, or percentile distribution) to communicate information about the data.
- Describe the effect of sample size and population size on the validity of predictions from a set of data.
- Design a fair game and provide a justification for fairness.
- Describe the differences between the theoretical probability of simple events and the experimental probability from simulations.
- Apply basic counting principles to describe simple events, and compute probabilities of events with outcomes that are not equally likely.

Geometry

The developmental sequence requires that students must know and be able to do all of the concepts and skills in grades 1-8, plus the following grade 9-10 standards:

Geometry

- Use deduction to establish the validity of geometric conjectures and to prove theorems in Euclidean geometry.
- Construct congruent and similar figures using a compass, straightedge, manipulatives, and other tools.
- Derive and apply properties of angles, arcs, chords, tangents, and secants to solve problems involving circles.
- Apply the geometry of linear graphs to parallel and perpendicular lines, and perform and interpret transformations of coordinates.
- Apply trigonometric ratios in right triangles to solve problems.
- Apply and interpret transformations on figures in the coordinate plane, e.g., translations, reflections, rotations, scale factors or size changes, and the results of successive transformations.
- Describe characteristics of discrete geometry (graph theory) and apply to the solution of problems.
- Identify and describe geometric patterns of change using recursive notation.

Measurement

- Relate changes in the measurement of one attribute of an object to changes in other attributes (e.g., how changing the radius or height of a cylinder affects its surface area or volume).
- Use simple tools (such as a clinometer) to solve indirect measurement problems.
- Apply the ratio of similarity, including the geometric mean and the relation-ships of special triangles, to the solution of problems.
- Describe the effects of rounding on measurements and on computed values from measurements.
- Apply formulas for surface area and volume of pyramids and cones.

Advanced Algebra/Precalculus

The developmental sequence requires that students must know and be able to do all of the concepts and skills in grades 1-10, plus the following grade 11-12 standards:

Number and Operations

- Describe the structure and properties of the real number system and relationships between the real number system and its various subsets.
- Define complex numbers and operate with them.
- Represent finite graphs using matrices and apply them to the solution of problems.
- Use combinatorics (e.g., permutations and combinations) to solve problems.

Patterns, Functions, and Algebra

- Describe and model phenomena using functions, including exponential, logarithmic, trigonometric, polynomial, rational, step, absolute value, and square root.
- Solve polynomial, exponential, logarithmic, and trigonometric equations and equations with rational expressions by symbolic (quadratic), graphical, and numerical methods. Apply each method when appropriate.
- Solve systems of equations and inequalities involving algebraic, exponential, logarithmic, and trigonometric expressions using symbolic, numeric, and graphical methods. Describe the relationships among the methods.
- Classify functions into families. Describe the effects of parameter changes on different representations of polynomial, rational, logarithmic, exponential, and trigonometric functions.
- Identify maximum and minimum values of functions and use them in applications.
- Perform operations on functions, including compositions. Find inverses of functions.
- Model real-world phenomena involving growth, decay, and periodic processes.
- Identify arithmetic and geometric sequences and series and their properties. Solve problems, including finding the n^{th} term recursively and explicitly.
- Define linear, exponential, and quadratic functions recursively and be able to find the closed forms.

Geometry

- Derive and apply trigonometric identities and the laws of sines and cosines.
- Use transformations and coordinate geometry to represent real-world and mathematical situations.
- Use vector geometry to solve problems. Describe addition of vectors and scalar multiplication both symbolically and pictorially. Use vector methods to obtain geometric results.
- Relate geometric and algebraic representations of curves, including conic sections.

Measurement

- Describe the relationship between degree and radian measures, and use radian measure in the solution of problems, e.g., angular velocity and acceleration.
- Use dimensional analysis for unit conversion and to confirm that expressions and equations make sense.

Data Analysis, Statistics, and Probability

- Describe a set of frequency distribution data by spread, skewness, symmetry, number of modes, or other characteristics.
- Apply regression results and curve fitting to make predictions from data.
- Use measures of spread of a set of data (variance, standard deviation) to solve problems.
- Design surveys and apply random sampling techniques to avoid bias in the data collected.
- Use simulations (e.g., random number tables, random functions, and area models) to determine experimental probabilities.
- Apply uniform, normal, and binomial distributions to the solutions of problems.

Appendix I:
Learning Standards by Grade Span

The grid on the next several pages shows the scope of learning standards and facilitates the identification of connections among the strands within each grade span. To design an effective mathematics program, it is important to examine the PreK-12 developmental sequence within each strand.

Number and Operations	Patterns, Functions, and Algebra	Geometry	Measurement	Data Analysis, Statistics, and Probability
GRADES PK – K				
• Count by ones to at least 100. • Match quantities up to at least 20 with numerals. • Compare sets of up to at least 20 concrete objects using appropriate language (e.g., more than, fewer than, same number of, one more than), and order numbers. • Model and solve addition sums to 10 and subtraction situations using objects and drawings. • Identify positions of objects in sequences (e.g., first, second). • Estimate the number of objects in a group.	• Sort and classify objects by color, shape, size, and number. • Reproduce, describe, extend, and create color, shape, number, and letter repeating patterns.	• Name, describe, and draw simple two-dimensional shapes, and identify shapes that have been rotated, reflected, or enlarged. • Describe attributes of two-dimensional shapes, e.g., number of sides, number of corners. • Name and describe three-dimensional concrete objects; compare attributes, e.g., number of faces, shape of faces. • Identify positions of objects in space, and use appropriate language (e.g., beside, inside, next to, above, below, nearer, farther) to describe and compare their relative positions.	• Use nonstandard and standard units to measure length, area, weight, and capacity. • Compare lengths, weights, capacities, and temperature using appropriate language, e.g., longer, taller, shorter, same length; heavier, lighter, same weight; holds more, holds less, holds the same amount; warmer, cooler, same temperature. • Identify hour and half-hour times using analog and digital clocks.	• Represent data using concrete models.

Number and Operations	Patterns, Functions, and Algebra	Geometry	Measurement	Data Analysis, Statistics, and Probability
GRADES 1 – 2				
• Identify and distinguish among multiple uses of numbers, including cardinal (to tell how many) and ordinal (to tell which one) numbers, and numbers as labels and as measurements. • Name and write (in numerals and words) whole numbers to at least thousands, identify the values of the digits, and order the numbers. • Use concrete materials to model addition and subtraction. • Describe various meanings of addition and subtraction, and the relationship between the two operations. • Know addition and subtraction facts (sums to 20), and use them to solve problems.	• Reproduce, describe, extend, and create simple rhythmic, shape, size, number, and color repeating patterns. • Describe and create addition and subtraction number patterns. • Write number sentences to represent mathematical relationships in real-world situations. • Construct and solve open sentences that have variables, e.g., □ + 7 = 10, □ + ❍ = 10. • Solve problems with input-output function machines and tables of data. • Describe functions related to trading, including coin trades (e.g., pennies for nickels, nickels for dimes) and measurement trades (e.g., cups for quarts).	• Identify, describe, draw, and compare two-dimensional shapes, including both polygonal and curved figures. • Describe attributes and parts of two- and three-dimensional shapes, e.g., number and length of sides, number of corners, edges, and faces. • Predict the results of putting shapes together and taking them apart. • Identify symmetry in two-dimensional shapes. • Recognize congruent shapes.	• Identify parts of the day (e.g., morning, afternoon, evening), week, month, and calendar. • Tell time at five-minute intervals on analog and digital clocks. • Identify pennies, nickels, dimes, quarters, $1, $5, $10, and $20 bills, and compute values of collections of coins and bills. • Make and use estimates of measurement, including time. • Measure common objects using metric and customary units of length and weight measurement, e.g., centimeter, inch, gram, and ounce; compare and order objects by some measure. • Select and use appropriate measurement tools, e.g., ruler, balance scale, and thermometer.	• Use interviews, surveys, and observations to gather data about themselves and their surroundings. • Organize, classify, and represent data using tallies, charts, tables, bar graphs, pictographs, and Venn diagrams; interpret the representations. • Formulate inferences (draw conclusions) and make hypotheses (educated guesses). • Decide which outcomes of experiments are most likely. • List and count the number of possible pairings of objects from two sets, e.g., how many different outfits can one make from a set of three shirts and a set of two skirts?

Number and Operations	Patterns, Functions, and Algebra	Geometry	Measurement	Data Analysis, Statistics, and Probability
GRADES 1 – 2				
• Find the value of a collection of coins less than $5.00 and different ways to represent an amount of money less than $5.00 with coins. • Identify common fractions (1/2, 1/3, 1/4) as parts of wholes, parts of groups, and numbers on a number line.				
GRADES 3 – 4				
• Name and write whole numbers to at least one million and decimals through hundredths, identify the values of the digits, and order the numbers. • Explain various meanings and models of multiplication and division of whole numbers and the inverse relationship between the two operations.	• Describe, extend, and create geometric and numeric patterns; make predictions and form generalizations about these patterns. • Identify and describe patterns created by multiples of whole numbers. • Use variables to represent unknowns in quantities that vary in expressions and in equations (mathematical sentences).	• Compare and analyze attributes of two- and three-dimensional geometric shapes, including diagonals. • Define and differentiate among various quadrilaterals, including squares, rectangles, rhombuses, parallelograms, and trapezoids. • Predict and validate the results of combining and partitioning two- and three-dimensional shapes.	• Identify time to the minute on analog and digital clocks. • Compute elapsed time, and make and interpret schedules. • Identify and use linear and square units of measurement appropriate for a particular situation. • Apply the formula for the area of a rectangle. • Identify angles as acute, right, or obtuse.	• Construct and interpret various representations of data sets, including tables, bar graphs, line graphs, and pictographs. • Classify outcomes as likely, unlikely, certain, and impossible by designing and conducting experiments using concrete objects, counters, number cubes, or coins.

Number and Operations	Patterns, Functions, and Algebra	Geometry	Measurement	Data Analysis, Statistics, and Probability
GRADES 3 – 4				
• Know multiplication and division facts through 12 x 12 and use them to solve problems. • Apply commutative, associative, and identity properties of operations on whole numbers. • Add, subtract, multiply (up to 3 digits by 2 digits), and divide with a single-digit divisor (with remainders). • Apply a variety of mental mathematics and estimation methods to problems involving up to 3-digit whole numbers and amounts of money to $1000. • Use models to relate fractions (1/2, 1/3, 1/4, 1/5, 1/6, 1/8, 1/10, and 1/12)to decimals, find equivalent fractions, and order fractions. • Use concrete objects to model operations with fractions and decimals.	• Describe procedures for finding values of variables in equations. • Solve problems involving proportional relationships, including unit pricing (e.g., four apples cost 80¢, so one apple costs 20¢) and map interpretation (e.g., one inch represents five miles, so two inches represents ten miles).	• Identify lines of symmetry in two-dimensional shapes. • Graph points in the first quadrant of the coordinate plane and identify the coordinates of points. Make and use coordinate maps to represent actual places. • Describe and draw intersecting, parallel, and perpendicular lines.		

Number and Operations	Patterns, Functions, and Algebra	Geometry	Measurement	Data Analysis, Statistics, and Probability
GRADES 3 – 4				
• Choose the appropriate operations (addition, subtraction, multiplication, division) to solve problems, including those involving money, and solve the problems.				
GRADES 5 – 6				
• Represent very large (billions, trillions, …) and very small (thousandths, millionths, …) positive numbers in various forms, including expanded notation with exponents. • Apply ratios and proportions to the solution of problems. • Describe relationships among fractions, decimals, and percents. • Solve problems involving addition, subtraction, multiplication, and division with whole numbers, fractions, and decimals (including percents). • Estimate results of computations with whole numbers, fractions, decimals, and percents.	• Extend and generalize numeric and geometric patterns. • Describe strategies for solving linear equations using concrete models, tables, graphs, and paper-pencil methods. • Produce and interpret graphical sketches that represent real events. • Represent real situations and mathematical relationships with concrete models, tables, graphs, and rules in words and with symbols. • Model situations with proportional relationships and solve problems.	• Identify properties of polygons. • Describe the relationship among points, lines, and planes. • Match three-dimensional objects and their two-dimensional representations (e.g., nets, projections, and perspective drawings). • Graph points on the Cartesian coordinate plane and identify coordinates of points. • Describe and perform transformations on shapes, e.g., translations, rotations, and reflections. • Describe and apply techniques for determining if two shapes are congruent.	• Solve problems involving proportional relationships and units of measurement, e.g., scale models, maps, and speed. • Find areas of triangles and parallelograms. Recognize that skewed versions of the same shape have the same area. • Apply the concept of perimeter to the solution of problems, and contrast it with the concept of area. • Identify, measure, describe, classify, and construct various angles and triangles. • Find sums of angle measures of polygons. • Describe the relationships of the radius, diameter, circumference, and area of a circle (e.g., $d = 2r$, $c/d = \pi$).	• Describe data sets using the concepts of median, mean, mode, maximum and minimum, and range. • Use tree diagrams and other models (e.g., lists and tables) to represent outcomes of experiments. Analyze the outcomes. • Construct and interpret stem-and-leaf plots and line plots. • Find probabilities of events with equally likely outcomes.

Number and Operations	Patterns, Functions, and Algebra	Geometry	Measurement	Data Analysis, Statistics, and Probability
GRADES 5 – 6				
• Compare and order whole numbers, decimals, and fractions. • Apply the Order of Operations, including parentheses. • Apply number theory concepts, including prime and composite numbers, prime factorization, greatest common factor, least common multiple, and divisibility to the solution of problems.			• Find volumes of rectangular prisms.	
GRADES 7 - 8				
• Estimate and compute with fractions, decimals, percents, and integers. • Represent numbers in scientific notation, and use them in problem situations. • Apply powers and roots to the solution of problems. Extend the Order of Operations to include exponents. • Define and apply frequently used irrational numbers, e.g., $\sqrt{2}, \pi$. •	• Describe, complete, extend, analyze, generalize, and create a wide variety of patterns, including iterative and recursive (e.g., Pascal's triangle), and linear and nonlinear functional relationships. • Use tables and graphs to compare linear, quadratic, and exponential growth patterns. • Demonstrate proficiency in solving linear equations and their applications using technology, graphs, and algebraic methods.	• Explain the meaning of the Pythagorean Theorem, and apply the theorem to the solution of problems. • Describe angles formed by intersecting lines and relationships among them. • Predict the results of transformations of the coordinate plane and draw the transformed figure. • Use technology, straight-edge, compass, or other tools to formulate and test conjectures and to make geometric constructions.	• Generalize the relationships between the number of sides and the sums of the angle measures of polygons. • Apply formulas and procedures for determining measures, including areas of polygons, and volumes and surface areas of prisms, cylinders, and spheres. • Use proportions to model and solve indirect measurement problems.	• Choose and apply appropriate measures of central tendency (mean, median, and mode) to represent a set of data. • Use tree diagrams, tables, and lists to describe sample spaces and to calculate probabilities of independent and dependent events. • Differentiate between continuous and discrete data and ways to represent them.

Number and Operations	Patterns, Functions, and Algebra	Geometry	Measurement	Data Analysis, Statistics, and Probability
GRADES 7 - 8				
• Describe conditions under which an estimate rather than an exact answer is appropriate. Apply in problem situations. • Apply number theory concepts, including relatively prime numbers, to the solution of problems. • Apply the Fundamental Counting Principle to the solution of problems. • Identify the properties of operations on integers and rational numbers, including closure, associativity, commutativity, distributivity, identity, and inverse.	• Represent and solve single-variable inequalities using symbols and graphs. • Identify the slope of a line as a constant rate of change from its table of values, equation, and graph. Apply the concept of slope to the solution of problems. • Explain and generalize how a change in one variable results in a change in another variable in functional relationships, e.g., $C = \pi d$, $A = \pi r2$, $A_{rectangle} = lw$, and $A_{triangle} = (2)bh$.	• Classify figures in terms of congruence and similarity, and apply these relationships to the solution of problems. • Apply networks to the solution of problems.	• Identify proportional relationships in similar plane figures and apply to the solution of problems. • Select and use an appropriate unit of measurement or scale.	• Make inferences about a characteristic of a population from a well constructed sample, e.g., capture-recapture. • Construct and interpret circle graphs. • Use box-and-whisker plots to represent data sets, and identify outliers.
GRADES 9 – 10				
• Apply operations with powers and roots, including fractional and negative exponents, to the solution of problems.	• Demonstrate facility in transforming polynomial expressions by rearranging and collecting terms, factoring, and applying the properties of exponents in order to solve problems.	• Use deduction to establish the validity of geometric conjectures and to prove theorems in Euclidean geometry. • Construct congruent and similar figures using a compass, straightedge, manipulatives, and other tools.	• Relate changes in the measurement of one attribute of an object to changes in other attributes (e.g., how changing the radius or height of a cylinder affects its surface area or volume).	• Represent data in a scatterplot. Use the scatterplot to make predictions. • Find a line of best fit from a set of data.

Number and Operations	Patterns, Functions, and Algebra	Geometry	Measurement	Data Analysis, Statistics, and Probability
GRADES 9 – 10				
• Use estimation to judge the reasonableness of results of computations and of solutions to problems involving real numbers.	• Identify problem situations that lead to linear, quadratic, or exponential equations and solve by applying appropriate graphical, tabular, or symbolic methods. Describe relationships among the methods. • Use algebraic and graphical methods to solve systems of linear equations and inequalities and describe relationships between different solution methods. • Use systems of equations or inequalities to represent mathematical relationships and to solve problems. • Describe similarities and differences among the families of linear, quadratic, and exponential functions using graphs, tables, formulas, and verbal descriptions. Describe the graphical significance of parameters. • Solve problems involving direct and inverse variation.	• Derive and apply properties of angles, arcs, chords, tangents, and secants to solve problems involving circles. • Apply the geometry of linear graphs to parallel and perpendicular lines, and perform and interpret transformations of coordinates. • Apply trigonometric ratios in right triangles to solve problems. • Apply and interpret transformations on figures in the coordinate plane, e.g., translations, reflections, rotations, scale factors or size changes, and the results of successive transformations. • Describe characteristics of discrete geometry (graph theory) and apply to the solution of problems. • Identify and describe geometric patterns of change using recursive notation.	• Use simple tools (such as a clinometer) to solve indirect measurement problems. • Apply the ratio of similarity, including the geometric mean and the relation-ships of special triangles, to the solution of problems. • Describe the effects of rounding on measurements and on computed values from measurements. • Apply formulas for surface area and volume of pyramids and cones.	• Select an appropriate graphical representation for a set of data and use appropriate statistics (e.g., mean, median, range, quartile, or percentile distribution) to communicate information about the data. • Describe the effect of sample size and population size on the validity of predictions from a set of data.Design a fair game and provide a justification for fairness. • Describe the differences between the theoretical probability of simple events and the experimental probability from simulations. • Apply basic counting principles to describe simple events, and compute probabilities of events with outcomes that are not equally likely.

Number and Operations	Patterns, Functions, and Algebra	Geometry	Measurement	Data Analysis, Statistics, and Probability
GRADES 11 – 12				
• Describe the structure and properties of the real number system and relationships between the real number system and its various subsets. • Define complex numbers and operate with them. • Represent finite graphs using matrices and apply them to the solution of problems. • Use combinatorics (e.g., permutations and combinations) to solve problems.	• Describe and model phenomena using functions, including exponential, logarithmic, trigonometric, polynomial, rational, step, absolute value, and square root. • Solve polynomial, exponential, logarithmic, and trigonometric equations and equations with rational expressions by symbolic (quadratic), graphical, and numerical methods. Apply each method when appropriate. • Solve systems of equations and inequalities involving algebraic, exponential, logarithmic, and trigonometric expressions using symbolic, numeric, and graphical methods. Describe the relationships among the methods.	• Derive and apply trigonometric identities and the laws of sines and cosines. • Use transformations and coordinate geometry to represent real-world and mathematical situations. • Use vector geometry to solve problems. Describe addition of vectors and scalar multiplication both symbolically and pictorially. Use vector methods to obtain geometric results. • Relate geometric and algebraic representations of curves, including conic sections.	• Describe the relationship between degree and radian measures, and use radian measure in the solution of problems, e.g., angular velocity and acceleration. • Use dimensional analysis for unit conversion and to confirm that expressions and equations make sense.	• Describe a set of frequency distribution data by spread, skewness, symmetry, number of modes, or other characteristics. • Apply regression results and curve fitting to make predictions from data. • Use measures of spread of a set of data (variance, standard deviation) to solve problems. • Design surveys and apply random sampling techniques to avoid bias in the data collected. • Use simulations (e.g., random number tables, random functions, and area models) to determine experimental probabilities. • Apply uniform, normal, and binomial distributions to the solutions of problems.

Number and Operations	Patterns, Functions, and Algebra	Geometry	Measurement	Data Analysis, Statistics, and Probability
GRADES 11 – 12				
	• Classify functions into families. Describe the effects of parameter changes on different representations of polynomial, rational, logarithmic, exponential, and trigonometric functions. • Identify maximum and minimum values of functions and use them in applications. • Perform operations on functions, including compositions. Find inverses of functions. • Model real-world phenomena involving growth, decay, and periodic processes. • Identify arithmetic and geometric sequences and series and their properties. Solve problems, including finding the n^{th} term recursively and explicitly. • Define linear, exponential, and quadratic functions recursively and to find the closed form expressions.			

Appendix II:
Instructional Technology

Instructional technologies like computers, the internet, graphing calculators, and other tools provide a means of data analysis and representation not possible until recent years. In the study of mathematics, instructional technologies allow students to display relationships graphically, display data in spreadsheets for numeric analysis, explore how changes in one variable affect another in graphical displays, and draw conclusions from ambiguous data. It is clear from pedagogical studies that students retain information and conclusions that they have derived themselves far more than those simply presented to them. For this reason, the emphasis in the framework standards is on students analyzing data and drawing their own conclusions. Instructional technologies are a significant aid to this process.

The instructional technology literacy competencies are a guide for districts to plan a systemic approach to ensure that all students learn technology skills appropriate to living and learning in the 21st century. These competencies are based on the National Educational Technology Standards Project in consultation with the U.S. Department of Education.

The instructional technology skills are divided into six broad categories: basic skills, social and ethical issues, productivity tools, communication tools, research tools, and problem solving tools. The competencies within each category are introduced, reinforced, and mastered by students throughout the PK-12 curriculum. They build upon each other in a logical progression. The category of *ethics and human issues,* for example, involves more than just teaching students how to use technology tools. It also involves discussions about the ethical dilemmas that arise when applying these tools. For example, students may conduct research on the internet, and at the same time discuss issues of plagiarism and fair use.

The sample performance indicators represent realistic, attainable activities that link mathematics standards to the competencies. They are examples of how students use these instructional technology skills to learn. Students should acquire basic technology skills by 8th grade. Students in grades 9-12 build on these skills as they use instructional technology to apply, demonstrate, generate, research, and evaluate ideas in each strand.

Technology integration requires training and support for teachers and students, and access to hardware and software. The Massachusetts Education Reform Act of 1993 calls for a statewide education technology plan (Mass Ed Online). To implement Mass Ed Online, Massachusetts has successfully undertaken multiple initiatives to increase the availability and use of technology in schools and classrooms. The instructional technology competencies are part of this effort to guide districts in their planning.

Instructional Technology in the Teaching of Mathematics

Category	Technology Competencies to be Acquired by Grade 8	In mathematics, students can:
Basic Skills and Operations	• Select and utilize appropriate applications (e.g., word processing programs, database, spreadsheet, multimedia, and web browser) for a variety of classroom projects. • Communicate about technology using appropriate and accurate terminology.	• Know when to use calculators, graphing calculators, spreadsheets, and databases, and know and use the terminology relevant to each type of technology.
Social, Ethical, and Human Issues	• Work cooperatively and collaboratively with peers when using technology in the classroom. • Identify ethical and legal behaviors when using technology in the classroom and describe personal consequences of inappropriate use. • Practice responsible use of technology systems and software.	• Always use complete and accurate citations when downloading information from the internet.
Technology Productivity Tools	• Use technology tools (e.g., word processing, database, spreadsheet, multimedia, web browsers) to increase productivity of individual and collaborative projects. • Use assistive technologies to remediate skill deficits when necessary. • Use technology tools and resources for managing and communicating personal or professional information (finances, schedules, correspondence).	• Use manipulatives and technology to model geometric shapes; use technology to perform transformations. (Geometry, grades 5-6, #5) • Gather real data and represent in a scatterplot by using technology. Use the scatterplot to make predictions. (Data Analysis, Statistics, and Probability, grades 9-10, #1)

Category	Technology Competencies to be Acquired by Grade 8	In mathematics, students can:
Technology Communi-cation Tools	• Use technology resources to communicate ideas and thoughts/stories: (e.g. word processing, e-mail, online discussions, Web environments) • Gather and analyze information using telecommunications. • Design, develop, publish and disseminate • Routinely and efficiently use online information resources to meet needs for collaboration, research, publications, communications, and productivity.	• Connect with classrooms in Massachusetts or other states or countries to share data and results of mathematical explorations. • Create mentor relationships via e-mail with retirees, college faculty, alumni, and professionals to complete classroom projects or mathematical challenges. • Research information and data using online resources such as "Ask a Mathematician."
Technology Research Tools	• Evaluate the accuracy, relevance, appropriateness, comprehensiveness, and bias of electronic information sources concerning real-world problems. • Select and apply technology tools for research.	• Research background information using on-line mathematics journals.
Technology Problem-Solving and Decision-Making Tools	• Use technology resources (simulations, charts) for problem solving. • Determine when technology is useful and select the appropriate tool(s) and technology resources to address a variety of tasks and problems. • Investigate and apply expert systems, intelligent agents, and simulations in real-world situations.	• Incorporate the use of technology such as CBL, CBR, software, and graphing calculators to investigate the graphical representation of functions and the properties of transformations. (Geometry, grades 9-10, Exploratory Concepts and Skills; Patterns, Functions, and Algebra, grades 11-12, #4 and 7)

Appendix III: Internet Resources

Curriculum and Assessment Resources

K-12 Mathematics Curriculum Center	www.edc.org/mcc
Mathematics Archives K-12 Teaching Material	archives.math.utk.edu/k12.html
Middle School Curriculum	showmecenter.missouri.edu
Curriculum Library Alignment and Sharing Project (CLASP)	www.massnetworks.org/clasp/clasp.html
Eisenhower National Clearinghouse for Mathematics and Science Education	www.enc.org
Massachusetts Department of Education	www.doe.mass.edu
PALMS Initiative	www.doe.mass.edu/palms
Regional Providers	www.doe.mass.edu/palms/PLM_regional.html
Mathematics Curriculum Framework 1995 and Public Comment Draft 1999	www.doe.mass.edu/doedocs/frameworks/

General Mathematics Education and Teacher Resources

Mathematical Association of America (MAA)	www.maa.org
Massachusetts Corporation for Educational Telecommunications (MCET) Mass Ed Online (MEOL) (mathematics subject area)	tom.mcet.edu/subjects/math.html
Quality Educators for Minorities Network	qemnetwork.qem.org
Voices of Girls in Math, Science and Technology	www.ael.org/nsf/
Technical Education Research Center (TERC)	web1.terc.edu
Math Forum	forum.swarthmore.edu
The Geometry Junkyard	www.ics.uci.edu/~eppstein/junkyard
National Council of Teachers of Mathematics	www.nctm.org
Math Teacher Link	www-cm.math.uiuc.edu/MathLink
Busy Teachers' Web Site	www.ceismc.gatech.edu/busyt/math.html
Global Schoolhouse	www.gsh.org
Ask ERIC Home Page	ericir.syr.edu
The Geometry Center	geom.umn.edu
Elementary School Teachers' Place	forum.swarthmore.edu/teachers/elem
American Mathematical Society	www.ams.org
National Science Foundation	www.nsf.gov

International Sites

Japanese Math Challenge	japanese-online.com/math/index.htm
NRICH Online Math Club	www.nrich.maths.org.uk
Third International Mathematics and Science Study (TIMSS)	ustimss.msu.edu

Math Contests

The Centre for Education in Mathematics and Computing	math.uwaterloo.ca:80/~cemc
Australian Mathematics Trust	www.amt.canberra.edu.au
Math League	www.mathleague.com

History of Mathematics

Women in Math Project	Darkwing.uoregon.edu/wmnmath/
History of Mathematics Home Page	aleph0.clarku.edu/~djoyce/mathhist/mathhist.htm
Mac Tutor History of Math Archive	www-groups.dcs.st-and.ac.uk:80/~history
Mathematical Quotations	math.furman.edu/~mwoodard/ascquotd.html

Number Theory

Math Fun with Spirolaterals	www.corona.bell.k12.ca.us/teach/swa/sept.html
Fantastic Fractals/Discover Fractals	library.advanced.org/12740/netscape/indes.html
The Centre for Experimental and Constructive Mathematics	cecm.sfu.ca

Puzzles and Games

Puzzle Archives	alabanza.com/kabacoff/Inter-Links/puzzles.html
Brain Teasers	www.eduplace.com/math/brain/index.html
Internet Math Challenge-U Idaho	www.uidaho.edu/LS/Math/imc
Interactive Mathematics Miscellany and Puzzles	www.cut-the-knot.com
Number and Word Puzzles	www1.tpgi.com.au/users/puzzles/

Appendix IV: Selected Problems and Activities

Number and Operations

Grades PreK – K
Refers to standards 1, 3, and 6
Have each child estimate the number of seeds in a slice of watermelon by inspection. Remove and count the seeds and compare the estimate to the count. Children then draw and color pictures of slices of watermelon, paste the seeds on their drawings, record the number of seeds, and compare their watermelon slices to tell who has more seeds.

Grades 1 – 2
Refers to standard 2
Use 8, 6, and 4.
Write the smallest three-digit number: _____
Write the greatest three-digit number: _____
Write other numbers using the same digits:

_____ _____ _____ _____

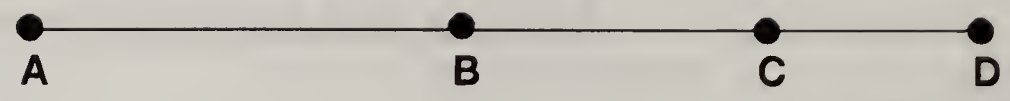
Refers to standard 7

Color ¼ of the small triangles.

Grades 3 – 4
Refers to standard 6
Hat - $4.52
Socks - $2.46
Sweater - $9.41
Scarf - $3.95

You have $20.
Do you have enough to buy all four items?
Estimate to find out.
Explain how you made your estimate.

Refers to standard 8
A to D is 16 miles.
A to C is 13 miles.
B to C is 6 miles.
A to B is ____ miles.



Grades 5 – 6
Refers to standards 2 and 4
The Hallett family is having its annual reunion. 192 family members plan to attend. Deb's favorite congo bar recipe makes 12 large bars, 18 medium bars, or 30 small bars. She needs to decide what sized congo bars to make and must increase the amount for each ingredient to serve her whole family.

Congo Bar Recipe
¾ cup sugar
2 eggs
1/2 cup brown sugar (packed)
2-1/4 cups flour (sifted)
1/8 tsp salt
1 tsp vanilla
1/4 cup nuts (optional)

What size bars would you suggest Deb make?
Help her calculate the amount of each ingredient she will need to make enough of your recommended size bars to feed her family.

Refers to standards 2 and 4
A. If X% of 12.5 is 37.5, then the percent is greater than 100.
B. If X% of 12.5 is 6.25, then the percent is less than 100.
Explain how the percent relates to the part and the whole in the above examples.
C. Find X if X% of 62 is 186.
D. Find X if X% of 62 is 15.5.

Grades 7 – 8
Refers to standard 1
You purchase one dollar of stock in Global Enterprises, Inc. On day 1, it rises 50% in value. On day 2, it falls 50% in value. On day 3, it rises 50% in value. On day 4, it falls 50% in value.
How much (to the nearest penny) is it worth at the end of day 4?

Refers to standard 6
If M is an odd number, then which of the following statements are true?
1. 3M is an odd number.
2. M^2 is an odd number.
3. $(M + 3)^2$ is an odd number.

Grades 9 – 10
Refers to standard 1
If you fold a piece of paper, you obtain one twice as thick, fold it again and you have one 4 times as thick, and so on. If you repeatedly fold a paper of thickness .01 cm (assuming you could do so), how many folds would it require for the thickness to reach from the earth to the moon, a distance of 400,000 km?

Refers to standard 3
Estimate each quantity and explain why your answer is reasonable.
1. In movies, you often see briefcases and suitcases allegedly filled with millions of dollars. About how many $1 bills into a briefcase?
2. Most major league baseball parks have a capacity of between 32,000 and 60,000 people. Concession sales are a significant part of each team's revenue. About how many hot dogs are sold at an average park during the 81 home-game season?

Grades 11 – 12
Refers to standard 4
Some services than involve electronic access require clients to choose a six-digit password. In an effort to increase security of the passwords, clients cannot use combinations that correspond to actual dates, nor can they use two identical digits in success, nor passwords with one digit appearing three or more times. How many "secure" passwords are available?

Patterns, Functions, and Algebra

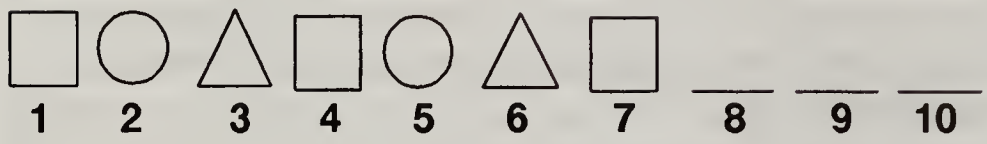
Grades PreK – K
Refers to standard 1
Give children a group of blocks of varying sizes, shapes, and colors. Have the children put blocks together that are the same color and talk about the shapes and sizes of those blocks. Follow the same procedure for size and shape.

Play the game "Mystery Block." Give clues about the block and ask for the solution.
Example: The mystery block is

- Red
- Large
- A square

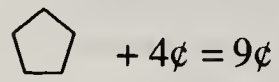
What is the mystery block?

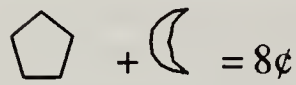
Refers to standard 2

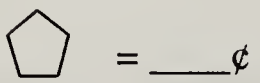


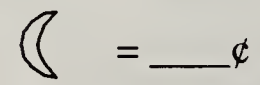
Identify the pattern. Draw 8, 9, and 10.

Grades 1 – 2
Refers to standard 5







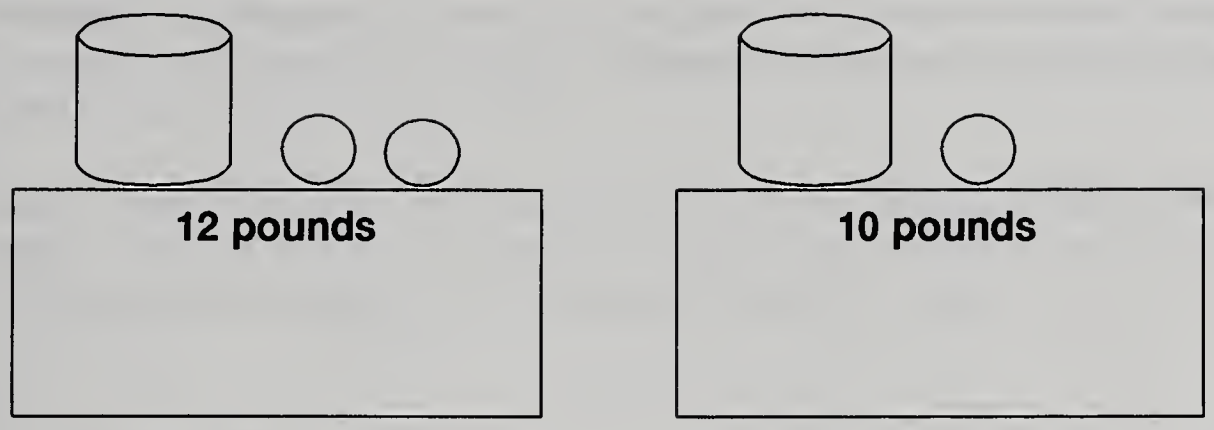


Refers to standard 6
Write <, = , or > in the [ ].

1. 11 nickels [ ] 5 dimes.
2. 2 quarters [ ] 55 pennies.
3. 4 dimes + 12 nickels [ ] 4 quarters.

Grades 3 – 4

Refers to standard 4



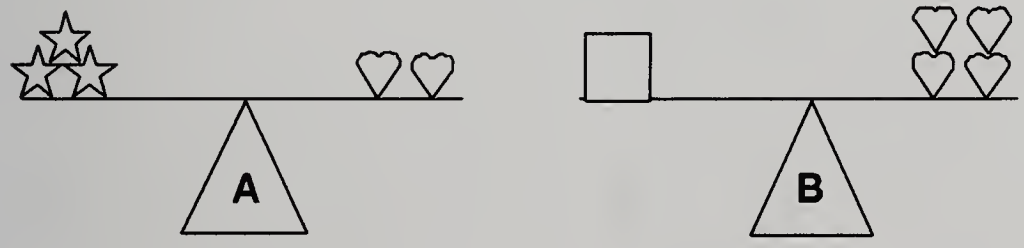
= ____ pounds

= ____ pounds

Tell how you solved the problem.

Refers to standard 5

Use the picture below to answer the questions.



1. How many stars will balance two squares?
2. How do you know?

Grades 5 – 6

Refers to standard 1

Triangles and trapezoids were used to make a pattern.

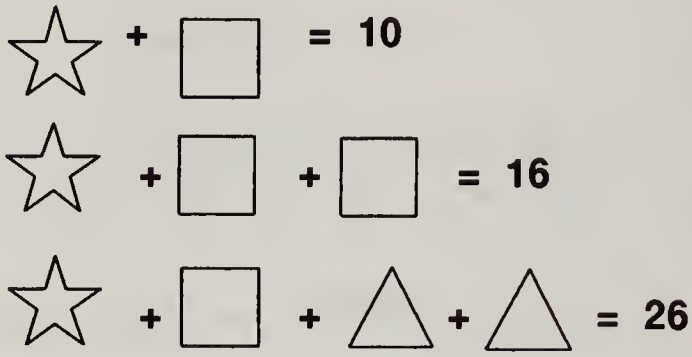
Level 1:

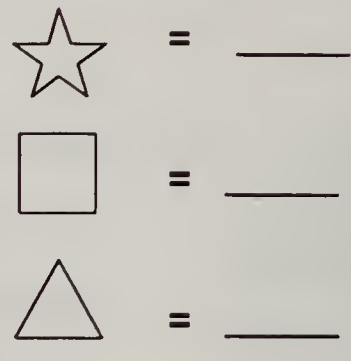
Level 2:

Level 3:

1. If the pattern above continues, how many black triangles are needed to build level 10? Explain how you know you are correct.
2. If the pattern above continues, how many white trapezoids are needed to build level 10? Explain how you know you are correct.

Refers to standard 2





Explain your solution strategy.

Grades 7 – 8

Refers to standard 1

```
          1
        1   1
      1   2   1
    1   3   3   1
  1   4   6   4   1
```

1. Construct the first 10 rows.
2. Identify different families of numbers in the diagonals.
3. Relate the numbers in the triangle to the row numbers.
4. Examine sums of rows. Relate row sums to the row numbers.
5. For each row, form two sums by adding every other number. Compare sums within and between rows. Describe the patterns that emerge and why they occur.
6. Describe how the triangle is developed recursively.

Grades 9 – 10

Refers to standards 2 and 6

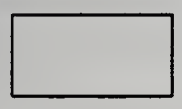
An ant moves across the top of a square at 4 mph, down the right side at 3 mph, across the bottom at 2 mph, and up the left side at 1 mph. What is the average speed of the ant, in mph?

Grades 11 – 12
Refers to standards 2 and 5
An open-top box is made from a single square piece of cardboard by cutting out a square from each corner and folding the sides.

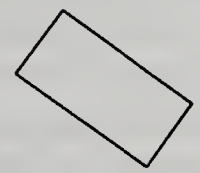
1. Sketch a graph that shows how the volume of the box depends on the ratio of the side length of the box to the side length of the original square piece of cardboard.
2. For what ratio does the box have the greatest volume?

Geometry

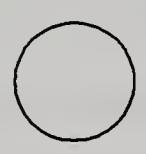
Grades PreK – K
Refers to standards 1 and 2
1. Count the sides.
2. Count the corners.
3. Tell the names of the shapes.

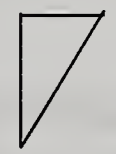
1. 

2. 

3. 

4. 

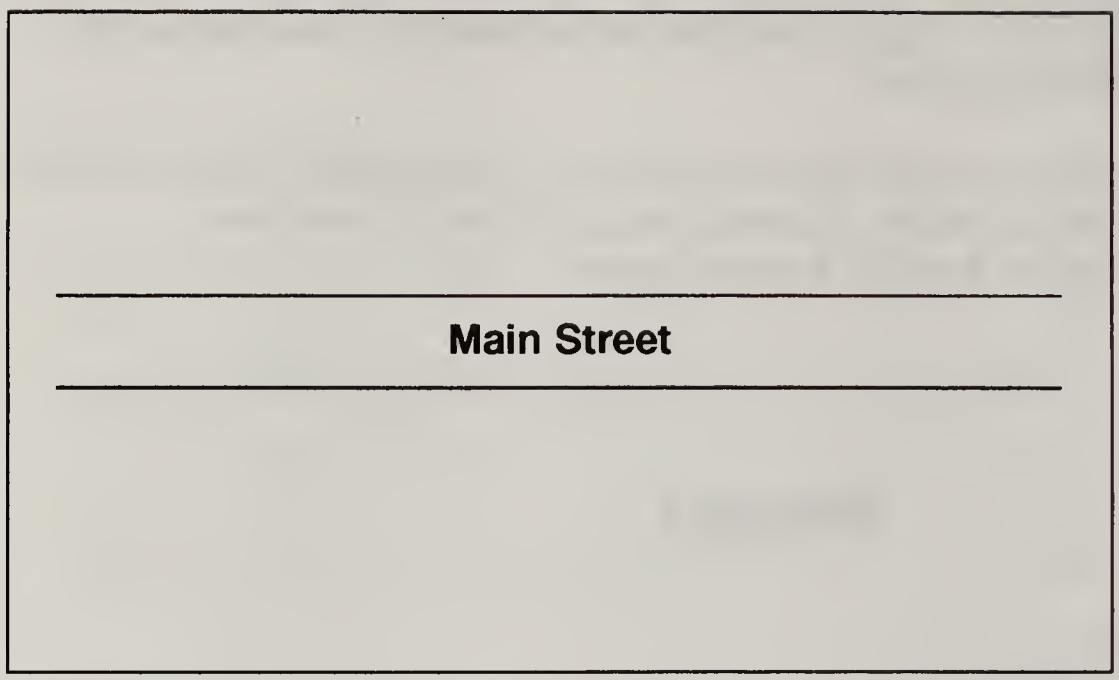
5. 

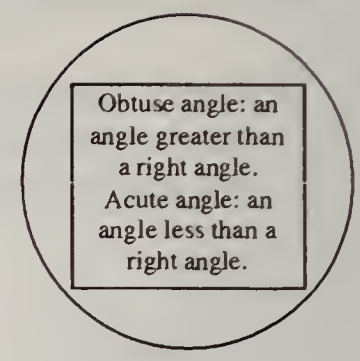
6. 

Grades 1 – 2
Refers to standards 2 and 5
Use geoblocks. For each block:
- Trace the faces.
- Identify the shape of each face.
- Tell the number of congruent faces.

Grades 3 – 4

Refers to standard 4 in grades 3-4





1. Draw Broadway Street *parallel* to Main Street. Write the name Broadway on this street.
2. Draw Birch Street *perpendicular* to main Street. Write the name Birch on this street.
3. Draw Walnut Street so that it *intersects* Main Street but is not perpendicular to Main Street. Write the name Walnut on this street.
4. Mark one *obtuse* angle on your drawing with the letter O.
5. Mark one *acute* angle on your drawing with the letter A.

Grades 5 – 6

Refers to standard 6

Storage boxes are cube shaped and measure 4 inches on an edge. How many of these storage boxes are needed for 300 small cubes, 2 inches on an edge?

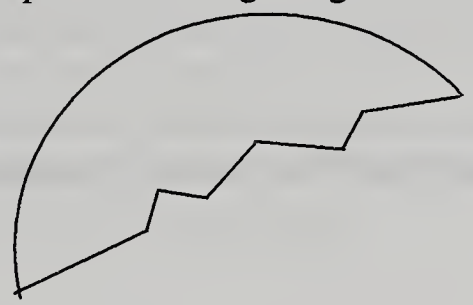
Grades 7 – 8

Refers to standard 2

A dog is tied to the stake with a 12 foot run. What is the maximum area that the dog can explore?

Grades 9 – 10
Refers to standard 2

Your grandmother's favorite glass-top table broke. There is only one big piece of the circular glass top left. You can buy a replacement top, but only if you know its diameter. Show how you can use a compass and straightedge to find the diameter of the old top.



Grades 11 – 12
Refers to standard 4
Concentric circles are drawn on a 1 cm^2 grid. The center of all of the circles is a point on the grid and the radii of the circles are 1 cm, 2 cm, 3 cm,…
Mark the point where a horizontal line is tangent to the 1 cm circle. Then mark the points where the line just above the first line intersects the 2 cm circle. Then mark the point where the next line intersects the 3 cm circle, and so on.

1. Identify the type on conic section on which lie all of the marked points.
2. Show that all the points lie on this type of curve. Use the geometric definition of this curve, or introduce a coordinate system and use the analytic definition.
3. Find the focus of this curve.
4. Identify some other collection of points formed by this pattern of lines and circles that lie on curves of the same kind.

Measurement

Grades PreK – K
Refers to standard 3
With the children's help, make a schedule of activities for the morning, recording times to the hour and half hour. Set the alarms of both a digital clock and an analog clock to ring at the start of each new activity. Call on children to read the clocks to verify that they match the times in the schedules.

Grades 1 – 2
Refers to standard 2
P N N P N N . . .
The patterns continues. There are 12 coins. How much money is that?

Grades 3 – 4
Refers to standard 4
Each □ is one square centimeter. What is the area of the letter?

Grades 5 – 6
Refers to standard 5
Determine the sum of the measures of the angles of an equilateral triangle, a square, a regular pentagon, a regular hexagon, and a regular octagon.

What is the relationship between the sum of the angle measures and the number of vertices in these polygons?

Grades 7 – 8
Refers to standard 5
At the end of every second mile of the Boston Marathon, a typical marathon runner takes about 4 ounces of water. Instead of drinking all of the water, the runner sips some of it and then throws the rest on his or her head or body to cool off.

1. At this rate, how many ounces of water would an average runner take in during an entire 26.2 mile marathon? Explain how you found your answer.
2. Suppose that all of the runners in the Boston Marathon behaved like the "typical" marathon runner described above. About how many gallons of water would have been used by the 40,700 runners in the 1996 Boston Marathon? Record each step you used to find your answer.

Grades 9 – 10
Refers to standard 2
Students replicate the experiment in which Eratosthenes calculated the circumference of the earth and got a remarkably good answer. They locate some schools roughly due north or south and connect with them through electronic mail. Students in each school agree that on a given day, at high noon, they will measure the shadow cast by a vertical stick on level ground. After sharing the measurements of the stick and the shadow, students use trigonometric ratios to determine the angle of the sun's rays. Using this information, along with the approximate distance between the schools, students use proportions to find an approximation of the earth's circumference. This example can be extended to sharing data with students from other states and countries.

Grades 11 – 12
Refers to standard 1
In one hour, the minute hand on a clock moves through a complete circle, and the hour hand moves through 1/12 of a circle. Through how many radians do the minute and the hour hand move between 1:00 p.m. and 6:45 p.m. on the same day?

Data Analysis, Statistics, and Probability

Grades PreK – K
Refers to standard 1
Children use chips to represent objects in the stories below in order to be able to count and compare the objects more easily.

Story 1
Mrs. Wigglybottom has large floppy hats. Put down one chip for each hat.
Mrs. Wigglybottom has a red hat with stars.
Mrs. Wigglybottom has a pink hat with flowers.
She has a green hat with ribbons.
She has a blue hat with feathers.
How many large floppy hats does she have?

Story 2
Mrs. Clippetyclop has animal hats. Put down one chip for each hat.
Mrs. Clippetyclop has a zebra hat with stripes.
She has a snake hat with scales.
She has a dog hat that barks.
How many animal hats does she have?

Who has more hats?

Grades 1 – 2
Refers to standard 6
José has lots of pennies, nickels, and dimes in his pocket. He takes out three coins and puts them on the table. How much money could be on the table? Make a list.

Grades 3 – 4
Refers to standard 2
There are two decks of cards. Deck 1 has the same number of clubs, spades, and diamonds. If you pick one card without looking, is it more likely that you will get a red card or a black card? Explain.

Deck 2 has the same number of clubs and diamonds. If you pick one without looking, is it more likely that you will get a red card or a black card? Explain.

Grades 5 – 6
Refers to standard 4
Joy has a bag of saltwater taffy of which 8 are lemon, 6 are peppermint, and 10 are licorice. She offers the bag to Marissa, who takes one piece of taffy without looking. What are the chances that it is peppermint?

Grades 7 – 8
Refers to standard 1
Shay took 5 tests, each worth 100 points. His average score was 85. What is the lowest score he could have received on one of the tests? Explain your answer.

Grades 9 – 10
Refers to standard 2
Use an almanac to find the winning times for the women's 400-meter freestyle swim for the Olympics from 1924-1984.
1. On graph paper, using 1920 as the base year, plot (year, time).
2. Construct a best-fit line.
3. What is the slope and what does it mean?
4. Write the equation of the line. Use the line to predict what the times might has been if the Olympics had been held in 1940 and 1944.

5. Is it reasonable to use this line to predict the winning time for the 1988 Summer Games. Why or why not?
6. Look us the winning time for the 400-meter freestyle swim in the 1988 Summer Games and compare it to the time predicted by the best-fit line.

Grades 11 – 12

Refers to standard 2

Research the changes in the number of cellular phones and personal computers in the United States between 1980 and 2000. First estimate, then use graphing calculators to decide whether the linear, quadratic, or exponential model is appropriate in each case. Compare growth rates and predict future changes in the use of each item. [The discussion should lead to topics in history and social studies related to growth and use of technology, including mathematical models to represent the changes.]

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Massachusetts Department of Education

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